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A DYNAMIC MODEL OF EDGEWORTH'S BARTER PROCESS

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Suppose that there are n commodities consumed by m consumers, and $x^i=\|x_{i1}, x_{i2}, \cdots\cdots\cdots\cdots, x_{in}\|'$, represents the quantities of consumption for the i-th consumer, $i=1, 2, \cdots\cdots, m$. Each consumer is assumed to have a preference relation $\underset{i}{\gtrsim}$ between vectors. Given any two vectors, x^i and y^i, either x^i is prefered to y^i ($x^i \underset{i}{\gtrsim} y^i$), or y^i is prefered to x^i ($x^i \underset{i}{\lesssim} y^i$), or x^i is indifferent to y^i ($x^i \underset{i}{\sim} y^i$). Where x^i is strictly preferred to y^i, we denote it by $x^i \underset{i}{\succ} y^i$. We postulate that the preference relation has the following properties:

(1) Irreflexivity: $x^i \sim x^i$ for any $x^i \geqq 0$.
(2) Transitivity: $x^i \underset{i}{\gtrsim} y^i$ and $y^i \underset{i}{\gtrsim} z^i$ implies $x^i \underset{i}{\gtrsim} z^i$.
(3) Monotonicity: $x^i \geqq y^i$ means $x^i \underset{i}{\gtrsim} y^i$.
(4) Insatiability: We can find y^i for any x^i such that $y^i \underset{i}{\gtrsim} x^i$ in the n-dimensional commodity space.
(5) Strong convexity: $x^i \underset{i}{\gtrsim} z^i$ or $x^i \underset{i}{\sim} z^i$ and $y^i \underset{i}{\gtrsim} z^i$ or $y^i \underset{i}{\sim} z^i$ with $y^i \neq x^i$ implies $(t \cdot x^i + (1-t) y^i) \underset{i}{\succ} z^i$ for any $0 < t < 1$, where t is a scalar.
(6) Continuity: $P=\{y^i | y^i \underset{i}{\gtrsim} x^i\}$ and $Q=\{y^i | x^i \underset{i}{\gtrsim} y^i\}$ are closed sets for any x^i.
(7) Semi-positivity of the individual consumer's resources: $x^i \geq 0$ for any i. If $x^r=0$ we can eliminate the r-th consumer from the discussion, because he can not exchange any quantities of commodities.
(8) If there is a numerical function $u_i(x^i)$ for $x^i \geqq 0$, such that $u_i(x^i) > u_i(y^i)$ if and only if $x^i \underset{i}{\succ} y^i$, and $u_i(x^i)=u_i(y^i)$ if and only if $x^i \underset{i}{\sim} y^i$, u_i is called a utility index of preference.

The barter process, discussed by Edgeworth, consists of successive barters between individuals according to their preferences and budget constraints.[1)]

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1) [2] pp. 316—319

In order to explain the bartering path, we will specify a so-called transaction rule among individuals regulating their behaviors over periods, at first.

The quantities exchanged and prices in period t will specify the exchange and prices in period $t+1$;

$$p(t+1)=\phi(p(t),\ X(t)) \tag{1}$$

$$x^i(t+1)=\psi_i(p(t),\ X(t)) \tag{2}$$

where

$$p(t)=\begin{Vmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{Vmatrix}$$ denotes the price vector in period t, and

$$X(t)=\begin{Vmatrix} x_{11}(t)x_{12}(t) & \cdots\cdots & x_{1m}(t) \\ x_{21}(t)x_{22}(t) & \cdots\cdots & x_{2m}(t) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ x_{n1}(t)x_{n2}(t) & \cdots\cdots & x_{nm}(t) \end{Vmatrix}$$

is the distribution matrix in period t, where x_{ij} denote the quantities of the i-th good distributed to the j-th agent. The mapping ϕ and ψ_i are continuous. Further we will give the definition of a Pareto optimum in this contex. If and only if there exists no distribution matrix $\dot{X}(t)$ such that

$$U(X(t))\geq U(X(t-1)) \quad (t=0,\ 1,\ 2,\ \cdots) \tag{3}$$

under the budget constraints

$$\sum_{i=1}^{n}\bar{p}_i'\cdot\bar{x}_{ij}=\sum_{i=1}^{n}p_i'(t)\cdot x_{ij}(t) \qquad \text{for } i=1,\ 2,\ \cdots,\ m, \tag{4}$$

where $\bar{x}_{ij}$ is the initial quantity of the j-th good kept by the i-th agent, $\bar{p}_i$ denote its related equilibrium price, then $X\ (t-1)$ is in a Pareto optimum. Here, $u_i(x^i(t))$ is the real valued mapping showing the preference index of $x^i(t)$, and $U(X(t))$ denotes the preference index vector in non-negative Euclidean m-dimensional vector space, $R_m{}^+$,

$$U(X(t))=\begin{Vmatrix} u_1(x^1(t)) \\ u_2(x^2(t)) \\ \vdots \\ u_m(x^m(t)) \end{Vmatrix}, \text{ where } \quad x^i(t)=\begin{Vmatrix} x_{1i}(t) \\ x_{2i}(t) \\ \vdots \\ x_{ni}(t) \end{Vmatrix}$$

for $i=1,\ 2,\ \cdots\cdots,\ m.$

THEOREM

The path from any semi-positive initial prices and distribution $\{\bar{p}, \bar{X}\}$ in Edgeworth's barter process converges to a Pareto optimum.

Proof

The commodity space for each individual is bounded and closed under the budget constraints (4) and non-negativity of x^i. The transaction will take place if and only if it will not make the individual worse off in terms of his preference relation:

$$U(X(t+1)) \geqq U(X(t)). \tag{5}$$

Since all of the distributions after transactions should satisfy the budget constraints and non-negativity of x^i, the commodity space itself is a compact set. According to the maximum value theorem[2)], the continuous mapping U has a maximum value,

$$\hat{U} = \max_{0 \leqq t} U(X(t)).$$

Let $\hat{X}$ be any limiting distribution of $X(t)$ as t tends to infinity; that is, for budget constraints (4), some sub-sequence $\{t_\nu\}$, $t \to \infty$ when ν converge to infinity:

$$\hat{X} = \lim_{\nu \to \infty} X(t_\nu).$$

When we consider the condition (5), we can get a sequence of preference indices as the following:

$$U(X(0)) \leqq U(X(1)) \leqq U(X(2)) \leqq \cdots\cdots\cdots\cdots\cdots \leqq \hat{U}.$$

At last, it will converge to $U(\hat{X})$ which ensures a muximum value of U, which is supported by the maximum value theorem.

Finally let us give the proof of uniqueness of the solution. We assume there are two Pareto optimum points, $\hat{X}^1$ and $\hat{X}^2$. By definition

$$\hat{U} = U(\hat{X}^1) = U(\hat{X}^2)$$

under the budget constraints $\sum_{i=1}^{n} \hat{p}_i \cdot \hat{x}_{ij}{}^1 \leqq \sum_{i=1}^{n} \bar{p}_i \cdot \bar{x}_{ij} = M_j$ and

$\sum_{i=1}^{n} \hat{p}_i \cdot \hat{x}_{ij}{}^2 \leqq \sum_{i=1}^{n} \bar{p}_i \cdot \bar{x}_{ij} = M_i$, for $i=1, 2, \cdots\cdots, m$. The point between $\hat{X}^1$

and $\hat{X}^2$ also satisfy the budget constraint, so

2) [3] p. 208, Theorem 2.

$$\sum_{i=1}^{n} \hat{p}_i \cdot \tfrac{1}{2}\hat{x}_{ij}^{1} + \sum_{i=1}^{n} \hat{p}_i \cdot \tfrac{1}{2}\hat{x}_{ij}^{2}$$

$$= \tfrac{1}{2}\sum_{i=1}^{n} \hat{p}_i \cdot \hat{x}_{ij}^{1} + \tfrac{1}{2}\sum_{i=1}^{n} \hat{p}_i \cdot \hat{x}_{ij}^{2}$$

$$\leqq M_i (i=1,\ 2,\ \cdots\cdots,\ \mathrm{m}),$$

where $\hat{p} = \|\hat{p}_1,\ \hat{p}_2,\ \cdots\cdots,\ \hat{p}_n\|' = \lim_{t\to\infty} p(t)$.

From assumption (5),

$$\frac{\hat{x}^1 + \hat{x}^2}{2} \succ \hat{x}^1 \sim \hat{x}^2.$$

Since $\hat{x}^1$ and $\hat{x}^2$ are in the Pareto optimum which ensures a maximum value of preference indices, a contradiction would result. So, the equilibrium point should be unique.

Q.E.D.

As explained by Edgeworth and Marshall, barter processes starting from any point may reach a point on the contract curve, and in turn, will converge to a Pareto optimum if and only if under the perfect competition the exchange goes on in an infinite number of periods.

REFERENCES

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(Aug. 31, 1968)