Article

ON THE CORE OF AN ECONOMY

— An Introductory Analysis —

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In the market, the manner in which prices are determined depends on the conditions of the market itself. Under perfect competition, large numbers of small households and firms determine demand and supply which in turn determine the equilibrium prices. A market in which supply is established by only one firm with an infinitive number of consumers provides knowledge about monopoly price. Between monopolistic and perfect competition, there are several stages of competition based on the number of economic agent in the market. Here, we shall discuss the situation in which there is a large number of small competitors forming a coalition.

1. THE CONCEPT OF CORE

The concept of coalition is developed in n-person game theory ($n \geqq 3$), but there are many unsolved problems in this theory eliminating from the fact that every situation is too complicated to build a general theory. Several methods of classifying games have been developed. The most important one among them is the classification by characteristic functions; that is, imputations and the core. Here, let us consider the real-valued function which has the following properties defining the characteristics of a general game; that is,

$v(\phi)=0$ where ϕ denotes the empty set. (1-1)

If R and S are any two disjoint subsets of players

$v(R \cup S) \geqq v(R)+v(S)$. (1-2)

It may be shown that in the case of a zero-sum game two additional conditions must be satisfied.

$v(I_n)=0$, where I_n is the set of all players. (1-3)

$v(S)=-v(-S)$, for all subsets S of I_n. (1-4)

If the function satisfies the conditions (1-1) and (1-2), we can call it a characteristic function of general games. Now, we denote by x_j the gain of the i-th player. As the result of a coalition, the i-th player will get the most advantageous gain, $\hat{x}_j$. So, we have

$$v(\{i\}) \leqq \hat{x}_i \text{ for } i \in I_n. \tag{1-5}$$

Since $v(I_n)$ represents the most that the players can obtain from the game by forming one grand coalition, it is impossible for $\sum_{i \in I_n} \hat{x}_i$ to exceed $v(I_n)$, so that

$$\sum_{i \in I_n} \hat{x}_i = v(I_n) \tag{1-6}$$

and $\hat{x}_i$ also satisfies Pareto optimality. Any n-tuple of real numbers which meets (1-5) and (1-6) is called an *impution* of the game with characteristic function v. We can then combine the two cases into one condition as the following:

$$v(S) \leqq \sum_{i \in s} x_i \text{ for } S \subseteq I_n. \tag{1-7}$$

The set of n-tuples satisfying condition (1-7) has been termed the *core* by Gillies.[1]

In the following paragraph, we will discuss the exchange market from the point of view of the *core* in which these n-tuples are included in any definition of equilibrium.

2. THE EDGEWORTH MODEL

As a simple model of an exchange economy, let us consider a market in which two consumers have certain initial quantities of two different commodities which they bring into the market, and in which production does not take place. The first consumer has $\bar{x}_{11}$ of the first commodity, $\bar{x}_{12}$ of the second commodity, and the second consumer brings $\bar{x}_{21}$ of the first commodity, $\bar{x}_{22}$ of the second commodity. In other words, the whole of the economy consume $\bar{x}_{11}+\bar{x}_{21}$ of the first commodity and $\bar{x}_{12}+\bar{x}_{22}$ of the second commodity. From this information we can get the well-known Edgeworth box diagram in Fig. 1. Here, I_1, I_2,$\cdots\cdots$are the indifference curves for the first consumer, I_1', I_2', $\cdots\cdots$denotes the curves for the second consumer, and A is the initial point where $o_1x_1=\bar{x}_{11}$, $o_1x_2=\bar{x}_{12}$, $o_2x_1'=\bar{x}_{21}$, and $o_2x_2'=\bar{x}_{22}$. The curve denoted by BC is the contract curve and the shaded area shows a more

1 D. B. Gillies. *Some Theorems on n-Person Games*. Ph. D. thesis, Department of Mathematics, Princeton University, Princeton, 1953.

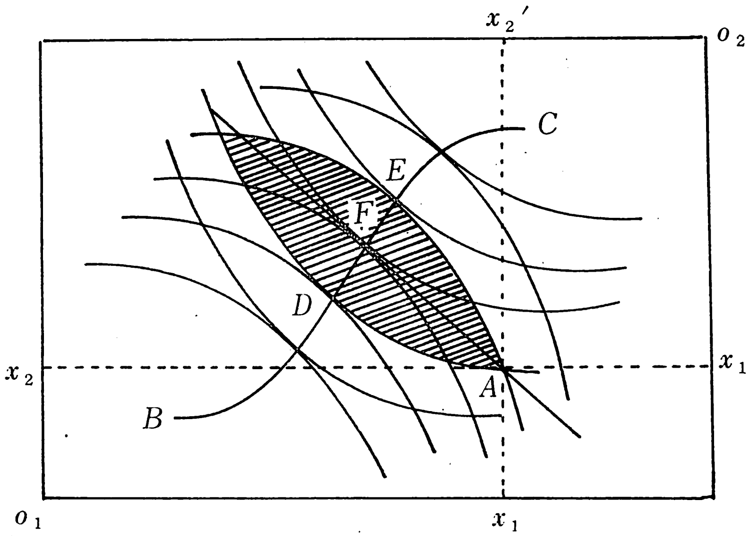

Fig. 1

preferable combination of commodities than the initial point A.

Then any point on DE is preferable to any other point not on it. If the slope of AF is a given price ratio between these commodities, F is a perfectly competitive equilibrium point. Edgeworth showed in his famous book, *Mathematical Psychics*, that the greater the number of consumers, the shorter will be the distance between D and E, and at the limit the segment DE converges to F as the number of consumers approaches infinity. This is perfect competition. In the following paragraph, in order to discuss this more precisely we shall make some assumptions concerning preferences and the market.

Suppose that there are n commodities consumed by m consumers, and $x^i=\|x_{i1}, x_{i2},\cdots\cdots x_{in}\|'$ represents the quantities of consumption for the i-th consumer, $i=1, 2,\cdots\cdots, m$. Each consumer is assumed to have a preference relation $\underset{i}{\gtrsim}$ between vectors. Given any two vectors, x^i and y^i either x^i is preferred to y^i $(x^i\underset{i}{\gtrsim}y^i)$, or y^i is preferred to x^i $(x^i\underset{i}{\lesssim}y^i)$, or neither is preferred to the other. In this last case, we shall also say that x^i is indifferent to y^i and write $x^i\underset{i}{\sim}y^i$. Where x^i is strictly preferred to y^i, we denote it by $x^i\underset{i}{\prec}y^i$. We postulate that the preference relation has the following properities:

(P-1) Irreflexivity: $x^i \underset{i}{\sim} x^i$ for any $x^i \geqq 0$.

(P-2) Transitivity: $x^i \underset{i}{\gtrsim} y^i$ and $y^i \underset{i}{\gtrsim} z^i$ implies $x^i \underset{i}{\gtrsim} z^i$.

(P-3) Monotonicity: $x^i \geq y^i$ means $x^i \underset{i}{\succ} y^i$.

(P-4) Insatiatiability: We can find y^i for any x^i such that $y^i \underset{i}{\succ} x^i$ in the n-dimensional commodity space.

(P-5) Strong conexity: $x^i \underset{i}{\gtrsim} z^i$ or $x^i \underset{i}{\sim} z^i$ and $y^i \underset{i}{\gtrsim} z^i$ or $y^i \underset{i}{\sim} z^i$ with $y^i \neq x^i$ implies $(t \cdot z^i + (1-t)y^i) \succ z^i$ for and $0 > t > 1$, where t is a scalar.

(P-6) Continuity: $P = \{y^i \mid y^i \underset{i}{\gtrsim} x^i\}$ and $Q = \{y^i \mid x^i \underset{i}{\gtrsim} y^i\}$ are closed sets for any x^i

(P-7) Semi-positivity of the individual consumer's resouces: $x^i \geq 0$ for any i. If $x^r = 0$ we can eliminate the r-th consumer from the discussion, because he can not exchange any quantities of commodities.

(P-8) If there is a numerical function $u_i\ (x^i)$ for $x^i \geqq 0$, such that $u_i(x^i) > u_i(y^i)$ if and only if $x^i \underset{i}{\succ} y^i$, and $u_i(x^i) = u_i(y^i)$ if and only if $x^i \underset{i}{\sim} y^i$, u_i is called a utility index of preference.

We also assume the following conditions for the market. Let us denote by Γ the set of all possible coalitions. So,

(M-1) $I_n \in \Gamma$.

(M-2) $A \in \Gamma$ and $B \in \Gamma$ implies $A/B \in \Gamma$. If A and B are two coalitions, the set of consumers in coalition A but not in coalition B is also a coalition.

(M-3) $A \in \Gamma$ and $B \in \Gamma$ implies that $A \cap B \in \Gamma$.

And then, from (P-7) we can easily define the core of an economy, that is, "the collection of all allocations of the total supply which cannot be blocked by any set S_i."[2] In other words, the core of an economy is the allocation of commodities for which there is no possibility of coalition, and there will be no advantage for individual consumers to make a coalition to redistribute their initial quantities of commodities.

Let us prove that the Edgeworth equilibrium point F in Fig. 1 is in the core. Let $\hat{x}^i$ denote the allocation of commodities when the coalition is entered into by the consumers in S_i. If p denote the price vector, we can get the following inequality;

2 G. Debreu and H. Scarf, "A Limit Theorem on the Core of an Economy," *International Economic Review*, Vol. 3. No. 3, 1962, p. 239.

$$p'\cdot\hat{x}^i \geqq p'\cdot x^i$$

when $\hat{x}^i \underset{i}{\gtrsim} x^i$ from the assumption (P-3). If x^i is strictly preferable to x^i, the inequality must hold, so

$$\sum_{i\in In} p'\cdot\hat{x}^i > \sum_{i\in In} p'\cdot x^i$$

which reduces to the following equation:

$$\sum_{i\in In} \hat{x}^i = \sum_{i\in In} \bar{x}^i$$

because *ex post* total supply identifies with *ex post* total demand. This implies that the allocation under perfect competition is in the core.

In the second stage we will treat the situation in which the set of $l\times k$ consumers are divided into subsets S_i, containing the same number of consumers ($i=1, 2,\cdots\cdots,l$): that is, the set is divided into l subsets according with the preference they have.

THEOREM 1

The allocation in the core is the same for all consumers having the same type of preferences, and the same vectors of initial resources.

Proof

Since I_n is divided into several subsets, S_i(i$=1, 2,\cdots\cdots l$), we can choose the a-th consumer in S_a and the b-th consumer in S_b according to their preference. In these subsets, we assume the allocations after exchange are the most unfavorable to the s-th consumer in S_a and the t-th consumer in S_b, and write

$$\left(\frac{1}{k}\sum_{a\in S_a} x_1{}^a, \frac{1}{k}\sum_{a\in S_a} x_2{}^a, \cdots\cdots\cdots\cdots \frac{1}{k}\sum_{a\in S_a} x_n{}^a\right) \underset{s}{\succ} x_s$$

and

$$\left(\frac{1}{k}\sum_{b\in S_b} x_1{}^b, \frac{1}{k}\sum_{b\in S_b} x_2{}^b, \cdots\cdots\cdots\cdots \frac{1}{k}\sum_{b\in S_b} x_n{}^b\right) \underset{t}{\succ} x_t$$

when initial resources are assigned unequally among the consumers in the same coalitions. For each commodity, *ex post* demand and supply are always identical, so that the following must hold:

$$\sum_{i\in K}\left(\frac{1}{k}\sum_{j\in S_i} x^j - \bar{x}^i\right) = 0,$$

where $K=\{1, 2,\cdots\cdots l\}$. If the a-th and b-th consumer form a coalition to assign their initial resources, they can get a better situation (assumption M-3, and P-6), so that an allocation in the core assigns the same commodity bundle to all consumers in the subset S_i.

Q.E.D.

THEOREM 2

Perfect competitive allocation in the core is on the Edgeworth equilibrium point.

Proof

The result of trading consists of an allocation of the total supply $\sum_{i \in I} \bar{x}^i$, and is therefore described by a collection of non-negative commodity bundles, x^i. Let Ω_i be the set of all $z^i = x^i - \bar{x}^i$, and $z^i + \bar{x}^i \succ_i x^i$, for each i. In the case of two consumers and two commodities, it can be shown in the Edgeworth box diagram, Fig. 2. Since Ω_i is an open convex set, Ω, the convex hull of the union of Ω_i is also an convex set. The non-negative linear combination of $z^i = \| z_{1i}, z_{2i} \cdots\cdots z_{ni} \|' \in \Omega_i$ ($i=1, 2\cdots\cdots$, m) represents

$$\alpha_1 z^1 + \alpha_2 z^2 + \alpha_3 z^3 + \cdots\cdots + \alpha_m z^m$$

where $\sum_{i=1}^{m} \alpha_i = 1, \ \alpha_i \geqq 0.$

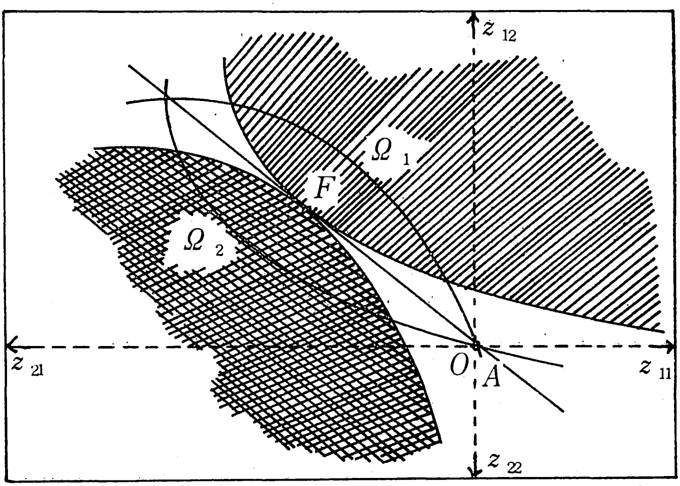

Fig. 2

At first, we are going to prove

$0 \notin \Omega$.

If we assume 0 is in Ω,

$0 = \alpha_1 z^1 + \alpha_2 z^2 + \alpha_3 z^3 + \cdots\cdots\cdots + \alpha_m z^m$.

We suppose α_i ($i=1, 2, \cdots\cdots w$) is a positive number, and the least integers, which are not larger than $k\alpha_1$, $k\alpha_2, \cdots\cdots$, $k\alpha_w$ where k is a given number, are denote $a_1{}^k$, $a_2{}^k, \cdots\cdots\cdots a_w{}^k$. The value of $z^{1k} = \left(\frac{{}^k\alpha_1}{a_1{}^k}\right) z^1, z^{2k} = \left(\frac{{}^k\alpha_2}{a_2{}^k}\right) z^2, \cdots\cdots, z^{wk} = \left(\frac{{}^k\alpha^w}{a_w{}^k}\right)$ converge to z^1, $z^2, \cdots\cdots$, z^w, where k is sufficiently large, so that $z^{1k} \in \Omega_1$, $z^{2k} \in \Omega_2, \cdots\cdots$, $z^{wk} \in \Omega_w$, because Ω_1 is an open set. Since

$$\sum_{i=1}^{w} a_i{}^k z^{ik} = k \sum_{i=1}^{w} \alpha_i z^i = 0$$

by the assumption, the coalition among consumers among sets Ω_1, Ω_2, $\cdots\cdots, \Omega_w$ bring some gain for each. That is inconsistent with the fact which the economy is in the core. So, we can reach the conclusion

$0 \notin \Omega$.

Q.E.D.

We can draw a hyperplane through the origin with a normal p, such as $p' \cdot z \geqq 0$ for $z \in \Omega$, that is

$p' \cdot z^i \geqq p' \cdot \bar{x}^i$.

But the total value of all commodities in terms of p must be equal to the value of initial commodities, because in this model production does not exist, so

$$\sum_{i \in I_n} p' \cdot x^i = \sum_{i \in I_n} p' . \bar{x}^i.$$

Therefore $p' \cdot x^i = p' \cdot \bar{x}^i$ for all $i \in I_n$. This shows the equilibrium point must on the buget line through the initial point A in Fig. 2, and must be on the boundary of the convex hull of the Ω_i for each i.

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