

**Equilibrium Indeterminacy under
Forward-Looking Interest Rate Rules**

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Abstract

Why is the rational expectations equilibrium locally indeterminate if the central bank raises the nominal interest rate too actively in response to a rise in expected inflation? This is because although the bank succeeds in stabilizing the expectations of the future economy, it allows the current economy to arbitrarily fluctuate. This indeterminacy expands in dimension as the forecast horizon of the rule becomes long.

Keywords: Active monetary policy; Sunspot equilibrium; Forecast horizon

JEL classification number: C62; E47; E52

1 Introduction

The literature has found that for the rational expectations equilibrium to be uniquely determined, the central bank should raise the nominal interest rate by more than one-for-one in response to a rise in the current inflation rate (i.e., *the Taylor principle*). On the other hand, Bernanke and Woodford [3] argue that the bank should follow the Taylor principle in response to the rate of *expected* inflation but *should not raise* the interest rate too actively. The reason for this is that the equilibrium also becomes indeterminate under too active a forward-looking interest rate rule¹. Moreover, Batini and Haldane [1] argue that if the forecast horizon of the rule is long, the forward-looking rule makes the economy fluctuating.

However, the related literature contains no work that explains the reason behind the forward-looking rule making the equilibrium indeterminate, although it does contain work in which the determinacy conditions on policy parameters have been analytically derived. Considering the fact that the effects of monetary policy are inclined to have a lag, it is essential that the central bank formulate a policy rule that is endowed with the forward-looking perspective. In addition, it is necessary to investigate the performance of a forward-looking rule with a long forecast horizon.

This paper presents two results. First, the emergence of indeterminacy under an active forward-looking rule results from the fact that although the central bank succeeds in stabilizing the expectations of the future economy, it allows the current economy to arbitrarily fluctuate. Second, as the forecast horizon of the forward-looking rule becomes long, there is an increase in the dimension of indeterminacy that makes the economy more fluctuating.

The next section shows a basic NK model and the conditions for equilibrium determinacy. Section 3 clarifies the reason for the existence of indeterminacy through

¹Similar arguments are offered by Clarida, Gali, Gertler [6] and Woodford [9, chapter 4].

simple assumptions. Section 4 simulates the impulse responses of stable sunspot equilibria that are possible under a forward-looking rule. The final section contains the main results.

2 The Model

We use a basic NK model to obtain the conditions for the determinate rational expectations equilibrium as described by Gali [5, chapter 3]².

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - 1/\sigma (i_t - E_t \pi_{t+1}), \quad (1)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \tilde{y}_t, \quad (2)$$

$$i_t = \phi E_t \pi_{t+1} + \varepsilon_t. \quad (3)$$

\tilde{y}_t denotes the output gap from its natural level in period t , π_t is the rate of inflation, and i_t is the nominal interest rate. E is the expectation operator. The last equation describes a forward-looking nominal interest rate rule where $\phi > 0$ is the policy parameter that represents the magnitude of the central bank's response to expected inflation. ε_t is a single exogenous fundamental shock that satisfies $E_{t-1} \varepsilon_t = 0$ for all t . $\sigma > 0$, $0 < \beta < 1$, $\kappa > 0$ are parameters.

Gali [5, chapter 4] provides the following sufficient and necessary condition for determinacy:

$$1 < \phi < \bar{\phi},$$

where $\bar{\phi} \equiv 1 + \frac{2\sigma(1+\beta)}{\kappa}$. This condition suggests that in addition to following the Taylor principle ($\phi > 1$), the central bank also *should not raise* the nominal interest rate too actively in response to a rise in the rate of expected inflation.

²Gali [5] introduces the natural rate r_t^n in (1) and the steady state nominal interest rate ρ in (3). We assume $r_t^n = \rho = 0$ to simplify the analysis. However, our analysis is unchanged.

3 Mechanism of Indeterminacy

To establish theoretical reasons for the indeterminacy that appears under a forward-looking rule, we consider equilibrium solutions under an active rule $\phi = +\infty$ as a simple case.

3.1 Under an active rule

Proposition 1 *For any value of fundamental shock ε in period t , if $\phi = +\infty$ under (3), the current equilibrium is indeterminate while the future equilibria are determinate as*

$$\begin{cases} \pi_t = \kappa \tilde{y}_t = -\frac{\kappa}{\sigma} i_t, \\ E_t \pi_{t+j} = E_t \tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 1. \end{cases} \quad (4)$$

(The proof is in Appendix A.)

This indeterminacy stems from the characteristic of the forward-looking rule. This rule implies that the central bank primarily focuses on stabilizing the expectations of future variables. Then, to the extent that a forward-looking rule is active, the bank allows the current economy to arbitrarily fluctuate while satisfying (4). As a result, the too active forward-looking rule makes the current equilibrium indeterminate³.

This result gives contrast to the equilibrium dynamics under a *current*-looking rule $i_t = \phi \pi_t$ instead of (3). Under the same assumptions in Proposition 1, the stable equilibrium is uniquely determined as

$$E_t \pi_{t+j} = E_t \tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } \forall j.$$

The derivation is in Appendix B.

³This characteristic resembles one of that of a passive rule ($\phi < 1$) in that the response of the bank to the fluctuations in the current variables is weak.

The current-looking rule differs from Proposition 1 in that it makes the current equilibrium determinate. This is because the central bank always focuses on stabilizing variables in the same period. Then the current economy as well as the future economy is uniquely determined only if $\phi > 1$.

As our simulations will show later, the mechanism shown in Proposition 1 works under the forward-looking rule of $\bar{\phi} < \phi < +\infty$ in general.

3.2 A long forecast horizon

The implication of Proposition 1 can be applied toward understanding the economy in which the central bank responds to the expected inflation of a *long forecast horizon*. As a simple example, we consider the following rule instead of (3):

$$i_t = \phi E_t \pi_{t+2} + \varepsilon_t. \quad (5)$$

Lemma 1 *For any value of fundamental shock ε in period t , if $\phi = +\infty$ under (5), the equilibrium solution has a two-dimensional indeterminacy. (The proof is in Appendix C.)*

The intuition is similar. Under this rule, the central bank gives top priority to stabilizing the expectations of the future variables from period $t+2$ onward. Then, to the extent that a forward-looking rule is active, the bank allows the current economy and that of the next period to arbitrarily fluctuate. This leads to a two-dimensional indeterminacy.

Lemma 1 is easily generalized as follows.

Proposition 2 *For any value of fundamental shock ε in period t , if $\phi = \infty$ under $i_t = \phi E_t \pi_{t+j}$ for $j \geq 0$, the equilibrium solution has j -dimensional indeterminacy. (The proof is similar to that of Lemma 1.)*

Batini and Haldane [1] argue that an inflation forecast rule with a long forecast horizon risks macroeconomic instability. However, their model is a reduced-form model, and thus, they do not provide any theoretical reason for the instability. Proposition 2 suggests that the instability is amplified by the increase in the dimension of indeterminacy.

4 Simulation

We simulate stable sunspot equilibrium dynamics to show that the fluctuations in the current variables are amplified to the extent that the central bank makes it active to respond to expected inflation. The values of our parameters are taken from Gali [5, p.51]: $\sigma = 1$, $\kappa = 0.1275$, $\beta = 0.99$, and then $\bar{\phi} \approx 32.2157$. In response to a sunspot shock in period 0 that generates a 0.1% rise in the expected inflation rate in period $t + 1$, we simulate sunspot dynamics under $\phi = \{32.5, 40, 50\}$. This sunspot shock abstracts a situation in which, for example, households happen to expect expansions of the future economy. The methodology applied to calculate sunspot equilibria is taken from Sims [8] and Lubik and Schorfheide [7]⁴.

Figure 1 shows the sunspot impulse responses. By assumption, the sunspot shock always raises the inflation rate in period $t + 1$ by 0.1%. Sunspot equilibria under $\phi > \bar{\phi}$ are oscillatory and converging to the steady state. The oscillatory convergence under $\phi > \bar{\phi}$ originates from the fact that in response to an increase in expected inflation above the steady state, the bank raises the current nominal rate so actively that the current variables drop below the steady state. Thus, the variables in the current and next periods continue to have signs opposite to each

⁴Sims [8] generalizes the methodology of Blanchard and Kahn [4] to obtain a solution of the rational expectations equilibrium. Lubik and Schorfheide [7] show a methodology to obtain the impulse responses of a sunspot equilibrium using Sims' methodology. The details are in Appendix D.

other around the steady state⁵⁶.

The magnitude of ϕ has a positive effect on the future economy and a negative one on the current economy. As ϕ becomes large, future variables get stabilized while current variables fluctuate. This is in keeping with the implication of Proposition 1 in that the central bank succeeds in stabilizing the future economy on the one hand and leaves the current economy to arbitrarily fluctuate on the other⁷.

5 Conclusion

In this paper, we have investigated the reason for the equilibrium being indeterminate when a forward-looking rule is too active. We also consider the reason for the economic instability when the forecast horizon of the rule becomes long.

The reason for the former is that if an active forward-looking rule is adopted, the central bank primarily focuses on stabilizing the expectations of future variables on the one hand and allows the current variables to arbitrarily fluctuate due to a sunspot shock on the other. The reason for the latter is that indeterminacy expands in dimension as the central bank focuses on stabilizing the expectations of inflation in the distant future.

⁵This mechanism is inherently equivalent to that of the sunspot equilibria under $\phi < 1$ that smoothly converge to the steady state, as simulated by Lubik and Schorfheide [7]. The smooth convergence under $\phi < 1$ is due to the fact that in response to a rise in the expected inflation, the central bank raises the current nominal rate less than one-for-one. Then, the positive sunspot shock in the expected inflation reduces the real interest rate, and the economy is modestly adjusted toward the steady state.

⁶If ϕ reaches unity or $\bar{\phi}$, sunspot responses stop converging. The equilibrium solutions are shown in Appendix E and F, respectively. If ϕ exceeds these values, the sunspot responses explode, leaving only fundamental responses in the neighborhood of the steady state.

⁷Note that this oscillatory dynamic is different from the *limit cycle* found by Benhabib, Schmitt-Grohe, and Uribe [2]. They show the possibility of *global indeterminacy* under a backward-looking rule.

Appendix

A Proof of Proposition 1

Suppose $\phi = +\infty$. For any value of fundamental shock ε in period t , (3) leads to $E_t\pi_{t+1} = 0$. This is applied to the expectations in period $t + 1$ onward. Then,

$$E_t\pi_{t+j} = 0 \text{ for } j \geq 1. \quad (6)$$

(2) and (6) derive $\pi_t = \kappa\tilde{y}_t$. This is applied to the expectations in period $t + 1$ onward. Then, from (6),

$$E_t\tilde{y}_{t+j} = \begin{cases} \pi_t/\kappa & \text{for } j = 0 \\ 0 & \text{for } j \geq 1 \end{cases}. \quad (7)$$

In addition, (1), (6), and (7) provide

$$E_t i_{t+j} = \begin{cases} -\sigma\tilde{y}_t & \text{for } j = 0 \\ 0 & \text{for } j \geq 1 \end{cases}.$$

i_t has an arbitrary value due to a sunspot shock in period t that is irrespective of fundamental shock ε .

To summarize, for any value of fundamental shock ε in period t , if $\phi = \infty$ under a forward-looking rule $i_t = \phi E_t\pi_{t+1}$, the equilibrium is

$$\begin{cases} \pi_t = \kappa\tilde{y}_t = -\frac{\kappa}{\sigma}i_t, \\ E_t\pi_{t+j} = E_t\tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 1. \end{cases}$$

B Equilibrium under $i_t = \infty \cdot \pi_t$

Suppose $\phi = +\infty$. For any value of ε in period t , (3) leads to

$$E_t\pi_{t+j} = 0 \text{ for } \forall j. \quad (8)$$

Next, (2) and (8) provide

$$E_t y_{t+j} = 0 \text{ for } \forall j. \quad (9)$$

However, if the economy satisfies (8) and (9), (1) indicates that the nominal interest rate must satisfy

$$E_t i_{t+j} = 0 \text{ for } \forall j.$$

In summary, for any value of fundamental shock ε in period t , if $\phi = \infty$ under a current-looking rule $i_t = \phi\pi_t$, the equilibrium is

$$E_t \pi_{t+j} = E_t \tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } \forall j.$$

C Proof of Lemma 1

Suppose $\phi = +\infty$. For any value of fundamental shock ε in period t , (5) leads to $E_t \pi_{t+2} = 0$. This is applied to the expectations in period $t+2$ onward. Then,

$$E_t \pi_{t+j} = 0 \text{ for } j \geq 2. \quad (10)$$

(2) and (10) give $E_t \pi_{t+1} = \kappa E_t \tilde{y}_{t+1}$. This is applied to the expectations in period $t+2$ onward. Then, from (10),

$$E_t \tilde{y}_{t+j} = \begin{cases} E_t \pi_{t+1} / \kappa & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases}. \quad (11)$$

In addition, (1), (10), and (11) provide

$$E_t i_{t+j} = \begin{cases} -\sigma E_t \tilde{y}_{t+1} & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases}.$$

Summarizing the solutions from period $t+1$ onward, for any value of fundamental shock ε in period t , if $\phi = \infty$ under $i_t = \phi E_t \pi_{t+2}$, the equilibrium is

$$\begin{cases} E_t \pi_{t+1} = \kappa E_t \tilde{y}_{t+1} = -\frac{\kappa}{\sigma} E_t i_{t+1}, \\ E_t \pi_{t+j} = E_t \tilde{y}_{t+j} = E_t i_{t+j} = 0 \text{ for } j \geq 2. \end{cases}$$

$E_t i_{t+1}$ has an arbitrary value due to a sunspot shock in period t that is irrespective of fundamental shock ε . This means that the equilibrium solution in period $t+1$ has a one-dimensional indeterminacy.

Further, given the value of $E_t\tilde{y}_{t+1}$, the solution in period t is expressed as functions of i_t as

$$\begin{cases} \pi_t = \kappa \left(1 + \beta + \frac{\kappa}{\sigma}\right) E_t\tilde{y}_{t+1} - \frac{\kappa}{\sigma} i_t, \\ \tilde{y}_t = \left(1 + \frac{\kappa}{\sigma}\right) E_t\tilde{y}_{t+1} - \frac{1}{\sigma} i_t. \end{cases}$$

i_t also has an arbitrary value due to another sunspot shock in period t that is irrespective of fundamental shock ε . That is, the solution in period t has greater one-dimensional indeterminacy even if $E_t\tilde{y}_{t+1}$ is given.

Therefore, for any value of fundamental shock ε in period t , if $\phi = \infty$ under $i_t = \phi E_t\pi_{t+2}$, the equilibrium has a *two-dimensional* indeterminacy.

D Derivation of a Sunspot Equilibrium

Our system is described in the manner employed by Sims [8] as follows:

$$\Gamma_0 Y_t = \Gamma_1 Y_{t-1} + \Psi z_t + \Pi \eta_t,$$

$$\text{where } Y_t = \begin{bmatrix} \tilde{y}_t \\ \pi_t \\ E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix}, z_t = \varepsilon_t, \eta_t = \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & -1 & (\phi - 1)/\sigma \\ -\kappa & 1 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} -1/\sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where η_t^x represents the forecast error of variable x (that is, $x_t = E_{t-1}x_t + \eta_t^x$).

A solution to a fundamental equilibrium under determinacy is given through equation (44) of Sims [8], and this solution corresponds to the solution of a fundamental equilibrium under indeterminacy that Lubik and Schorfheide [7] define under the assumption of *Orthogonality*. A solution to a sunspot equilibrium under indeterminacy is the sum of the above fundamental solution and a forecast

error component $H \begin{bmatrix} Q_1 & -\Phi Q_2 \\ & 0 \end{bmatrix} \Pi \eta_t$, notations of which are taken from Sims [8]. These solutions are computed using a MATLAB program "gensys.m," which is given in Professor Sims' homepage.

E Equilibrium under $i_t = E_t \pi_{t+1}$

Suppose that a sunspot shock causes households to believe that the inflation rate is non-zero and permanently constant; for example, $E_t \pi_{t+j} = \pi \neq 0$ for $j \geq 0$. Then, if $\phi = 1$, the following smooth and non-converging sunspot equilibrium satisfies our equations:

$$\begin{aligned} E_t \pi_{t+j} &= \pi, \\ E_t \tilde{y}_{t+j} &= \frac{1-\beta}{\kappa} \pi, \\ E_t i_{t+j} &= \pi \quad \text{for } j \geq 0. \end{aligned}$$

F Equilibrium under $i_t = \bar{\phi} E_t \pi_{t+1}$

Suppose that a sunspot shock causes households to believe that the absolutes of variables are non-zero and constant and that the signs of variables change period by period; for example, $E_t \pi_{t+j} = (-1)^j \pi \neq 0$ for $j \geq 0$. Then, if $\phi = \bar{\phi} = 1 + \frac{2\sigma(1+\beta)}{\kappa}$, the following cyclical and non-converging sunspot equilibrium satisfies our equations:

$$\begin{aligned} \pi_{t+j} &= (-1)^j \pi, \\ \tilde{y}_{t+j} &= (-1)^j \frac{1+\beta}{\kappa} \pi, \\ i_{t+j} &= (-1)^j \left(1 + \frac{2\sigma(1+\beta)}{\kappa} \right) \pi \quad \text{for } j \geq 0. \end{aligned}$$

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Figure 1 Impulse Responses to a Sunspot Shock

