CRITICALITY OF SCHRÖDINGER FORMS AND LIOUVILLE-TYPE PROPERTY

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Abstract. Pinchover proved Liouville-type property for Schrödinger operators by applying the criticality theory. We extend in [27] the criticality theory to Schrödinger forms. Employing this result, we show Liouville-type property for Schrödinger forms with non-local part.

1. Introduction

In [30, Section 5], we considered a Liouville-type property for Schrödinger forms with non-local part. This paper is a continuation of it. Our aim is to extend a result of Pinchover [22] on Liouville-type property to more general Dirichlet forms with non-local part.

Let E be a locally compact separable metric space and $E_\Delta = E \cup \{\Delta\}$, the one point comactification of E. Let m be a positive Radon measure on E with full topological support. Let $X = (\mathbf{P}_x, X_t, \zeta)$ be an m-symmetric Hunt process. We assume that X is irreducible and strong Feller, in addition, the associated Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ is *regular*. Let μ^+ be a Radon measure in the local Kato class ($\mu^+ \in \mathcal{K}_{\rm loc}$) and $X^+ = (\mathbf{P}^+_x, X_t, \zeta)$ the subprocess of X by $e^{-A^{\mu^+}_t}$:

$$d\mathbf{P}^+_x = \exp(-A^{\mu^+}_t) \cdot d\mathbf{P}_x,$$

where $A^{\mu^+}_t$ is the positive continuous additive functional (PCAF for short) with Revuz measure μ^+. We suppose that X^+ is also strong Feller. Let μ^- be in a non-trivial measure in the Green-tight Kato class with respect to X^+ ($\mu^- \in \mathcal{K}_\infty(X^+)$ for short). For the definition $\mathcal{K}_\infty(X^+)$, see Definition 2.1. Suppose that μ^+ is also non-trivial when X is recurrent. In other words, X^+ is always supposed to be transient. Throughout this paper, we deal with measures $\mu = \mu^+ - \mu^- \in \mathcal{K}_{\rm loc} - \mathcal{K}_\infty(X^+)$.

For $\mu \in \mathcal{K}_{\rm loc} - \mathcal{K}_\infty(X^+)$, we define a Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ by

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) + \int_E \widetilde{u}^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^\mu)(= \mathcal{D}(\mathcal{E}) \cap L^2(E;\mu^+)),$$

where $\widetilde{u}$ is a quasi-continuous version of u. Denote by $\mathcal{L}^\mu$ the self-adjoint operator associated with the closed symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, $(-\mathcal{L}^\mu u, v)_m =$

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$\mathcal{E}^\mu(u,v)$. We see that the semi-group $p_t^\mu = \exp(t\mathcal{L}^\mu)$ is expressed by the Feynman-Kac semi-group

$$p_t^\mu f(x) = \mathbf{E}_x\left(e^{-A_t^\mu} f(X_t)\right), \quad A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}.$$

We denote by $\mathcal{D}_{\rm loc}(\mathcal{E})$ the set of functions locally in $\mathcal{D}(\mathcal{E})$. Each function u in $\mathcal{D}_{\rm loc}(\mathcal{E})$ admits a quasi-continuous version $\widetilde{u}$. In the sequel, we write again u for $\widetilde{u}$ simply. Define the measure on E by

$$\mu_{\langle u\rangle}^j(B) = \iint_{B\times E}(u(x)-u(y))^2 J(dx,dy), \quad B\in\mathscr{B}(E),$$

where J is the jumping measure in the Beurling-Deny formula ([11, Theorem 3.2.1]) and $\mathscr{B}(E)$ is the set of Borel subsets of E. We introduce a subspace of $\mathcal{D}_{\rm loc}(\mathcal{E})$:

$$\mathcal{D}_{\rm loc}^\dagger(\mathcal{E}) = \left\{u\in\mathcal{D}_{\rm loc}(\mathcal{E}) \;\middle|\; \mu_{\langle u\rangle}^j \text{ is a Radon measure on } E\right\}$$

Kuwae [14] introduces the space $\mathcal{D}_{\rm loc}^\dagger(\mathcal{E})$ and shows that a bounded function in $\mathcal{D}_{\rm loc}(\mathcal{E})$ belongs to $\mathcal{D}_{\rm loc}^\dagger(\mathcal{E})$ and $\mathcal{E}(h,\varphi)$ is well-defined for $h\in\mathcal{D}_{\rm loc}^\dagger(\mathcal{E})$ and $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(X)$, even if $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ contains the jumping part (see also [10]).

We define the function space of p_t^μ-excessive functions by

$$\mathcal{H}_+^{\rm exc}(\mu) = \{h \mid h \text{ is quasi-continuous}, \ h>0 \text{ q.e.}, \ p_t^\mu h\le h \text{ q.e.}\}. \tag{1.1}$$

We see from [16] that if $h\in\mathcal{H}_+^{\rm exc}(\mu)$, then the bilinear form $(\mathcal{E}^{\mu,h},\mathcal{D}(\mathcal{E}^{\mu,h}))$ on $L^2(E;h^2m)$ defined by

$$\begin{cases}\mathcal{E}^{\mu,h}(u,u) = \mathcal{E}^\mu(hu,hu)\\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u\in L^2(E;h^2m)\mid hu\in\mathcal{D}(\mathcal{E}^\mu)\}\end{cases} \tag{1.2}$$

is a *quasi-regular Dirichlet form* on $L^2(E;h^2m)$. As a result, if $\mathcal{H}_+^{\rm exc}(\mu)$ is not empty, then $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite,

$$\mathcal{E}^\mu(u,u) = \mathcal{E}^{\mu,h}(u/h,u/h)\ge 0.$$

In the sequel, for a symmetric form $a(u,v)$ we simply write $a(u)$ for $a(u,u)$.

The subcriticality, criticality and superciriticality for the Schrödinger form $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ are defined as follows: $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is said to be *subcritical* (resp. *critical*) if $\mathcal{H}_+^{\rm exc}(\mu)$ is not empty and $(\mathcal{E}^{\mu,h},\mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient (resp. recurrent) for some $h\in\mathcal{H}_+^{\rm exc}(\mu)$. Besides, $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is said to be *supercritical* if $\mathcal{H}_+^{\rm exc}(\mu)$ is empty. We proved in [27] that these definitions are well-defined and obtained an analytic criteria for the criticality: Indeed, define

$$\lambda(\mu) = \inf\left\{\mathcal{E}^{\mu^+}(u) \;\middle|\; u\in\mathcal{D}(\mathcal{E}^\mu), \ \int_E u^2 d\mu^- = 1\right\}. \tag{1.3}$$

Then, $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is subcritical and critical if $\lambda(\mu)>1$ and $\lambda(\mu)=1$ respectively (Theorem 3.4 below). As an application of these results, we give a Liouville-type condition for a positive semi-definite Schrödinger form $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ being critical.

A function h on E is called a *solution* (*subsolution*, *supersolution*) to $\mathcal{L}^\mu u = 0$ if $h\in\mathcal{D}_{\rm loc}^\dagger(\mathcal{E})\cap L_{\rm loc}^\infty(E;m)$ and

$$\mathcal{E}^\mu(h,\varphi) = 0 \ (\le 0, \ \ge 0) \text{ for } \forall\varphi\in\mathcal{D}(\mathcal{E})\cap C_0^+(E).$$

Denote by $\mathcal{H}_+^{\rm sol}(\mu)$ (resp. $\mathcal{H}_+^{\rm sub}(\mu)$, $\mathcal{H}_+^{\rm sup}(\mu)$) the function space of positive solutions (resp. subsolution, supersolution). We construct elements in these space in the variational method, employing the fact that $(\mathcal{D}_e(\mathcal{E}^{\mu^+}), \mathcal{E}^{\mu^+})$ is compactly embedded in $L^2(E;\mu^-)$ ([29, Theorem 4.8]), where $\mathcal{D}_e(\mathcal{E}^{\mu^+})$ is the extended Dirichlet space of the transient Dirichlet form $(\mathcal{E}^{\mu^+}, \mathcal{D}_e(\mathcal{E}^{\mu^+}))$ (For the definition of extended Dirichlet space, see the first paragraph in Section 2.). This is the reason why we suppose $\mu^- \in \mathcal{K}_\infty(X^+)$. In this respect, our results do not implies the known ones for elliptic operators.

We show in Lemma 4.3 that if h is a non-negative supersolution, then $p_t^\mu h(x) \le h(x)$ quasi-everywhere (q.e. for short) x. In particular, $\mathcal{H}_+^{\rm sup}(\mu)$ is contained in $\mathcal{H}_+^{\rm exc}(\mu)$, which is proved in [27] when the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local and in [21] extended to general Dirichlet forms. In this paper we will give a probabilistic proof of Lemma 4.3. Indeed, using the stochastic calculus associated with regular Dirichlet forms, in particular, the *generalized Fukushima decomposition* due to Kuwae [15, Theorem 1.2] and the identification of its 0-energy part due to Miura [21, Corollary 3.3], we show Lemma 4.3. As a corollary of Lemma 4.3, we prove that a non-negative supersolution is positive q.e. or equal to zero q.e. (Theorem 4.4). The property is called the *strong maximum principle* and has been considered in [2], [5].

We would like to make a comment on the proof of Lemma 3.1. We apply the generalized Fukushima decomposition to the *resurrected Hunt process* $X^{\rm res}$ generated by the regular Dirichlet form $(\mathcal{E}^{\rm res}, \mathcal{D}(\mathcal{E}^{\rm res}))$ which is constructed by removing the killing part from $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ ([7, Theorem 5.2.17]). Since the killing measure of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is identified the Revuz measure corresponding to the dual predictable projection of $1_{\{X_{\zeta-}\neq\Delta,\, \zeta\le t\}}$ (cf. [11, Theorem 5.3.1]) and the paragraph below Theorem 5.3.1), removing the killing part from the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is equivalent to removing the *totally inaccessible* part from the life time ζ. As a result, the life time of $X^{\rm res}$ turns out to be a *predictable* stopping time and the original process X is the subprocess of $X^{\rm res}$ relative to the multiplicative functional $\exp(-A_t^k)$, where A_t^k is the positive continuous additive functional in the Revuz correspondence to th killing measure k in the Beurling-Deny formula. This obsevation make the proof of Lemma 4.3 easier.

We define a positive bilinear form by

$$\mathcal{E}^h(\varphi) = \frac{1}{2}\int_E h^2 d\mu^c_{\langle\varphi\rangle} + \iint_{E\times E\setminus d} h(x)h(y)(\varphi(x)-\varphi(y))^2 J(dx,dy), \tag{1.4}$$

where $\mu^c_{\langle\varphi\rangle}(=\mu^c_{\langle\varphi,\varphi\rangle})$ is the local part of energy measure of $\mu_{\langle\varphi\rangle}(=\mu_{\langle\varphi,\varphi\rangle})$, $u, v\in\mathcal{D}(\mathcal{E})$(cf. [11, p. 126]). We then see that for $h \in \mathcal{D}^\dagger_{\rm loc}(\mathcal{E})\cap L^\infty_{\rm loc}(E;m)$ and $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$, $\mathcal{E}^h(\varphi)$ is finite and

$$\mathcal{E}^\mu(h\varphi) = \mathcal{E}^h(\varphi) + \mathcal{E}^\mu(h, h\varphi^2). \tag{1.5}$$

Suppose $\mu^-\in\mathcal{K}_\infty(X^+)$. If $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical, then there exists a positive function h_0 in $\mathcal{D}_e(\mathcal{E}^{\mu^+})\cap C_b(E)$ such that $p_t^\mu h_0 = h_0$ and $\mathcal{E}^\mu(h_0,\varphi)=0$ for $\forall\varphi\in\mathcal{D}(\mathcal{E})$. Moreover, every $h\in\mathcal{H}_+^{\rm exc}(\mu)$ is written as $h = ch_0$, $c>0$. Hence, we see that if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical, then $\mathcal{E}^\mu(h\varphi)=\mathcal{E}^h(\varphi)$ for a locally bounded $h\in\mathcal{H}_+^{\rm exc}(\mu)$.

Define

$$\Lambda(\mu)=\inf\left\{\mathcal{E}^\mu(u)\,\Big|\,\int_E u^2dm=1\right\} \tag{1.6}$$

and suppose that $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite, i.e. $\Lambda(\mu)\geq 0$. We see that $\Lambda(\mu)\geq 0$ is equivalent to $\lambda(\mu)\geq 1$. Hence, if $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite, then it is subcritical or critical. We prove in Theorem 6.3 if there exists a non-trivial solution h to $\mathcal{L}^\mu u=0$ and a function ρ such that

(i) $|h|\leq\rho$
(ii) $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E})\cap C_0(E))$ is closable and its closure $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$ is a recurrent Dirichlet form on $L^2(E;m)$,

then $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is critical. By contraposition to this statement, we have a Liouville-type property that if $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is subcritical, that is, $\lambda(\mu)>1$, then a solution h satisfying the conditions (i), (ii) above is trivial, $h\equiv 0$ (Theorem 6.4 and Theorem 6.6). Moreover, a solution is not sign-changing if $\lambda(\mu)\geq 1$. For example, let us consider the classical case $\mathcal{L}^\mu=\Delta-\mu$. Let h be a solution to $\mathcal{L}^\mu u=0$. If there exists a function $\rho\in H^1_{\rm loc}(\mathbb{R}^d)$ such that $|h|<\rho$ and $\displaystyle\int_1^\infty\frac{rdr}{\int_{|x|\leq r}\rho^2dx}=\infty$, then h satisfies (ii) ([26, Theorem 3]) and not sign-changing, in particular, if $h=O(|x|^{1-\frac{d}{2}})$ as $|x|\to\infty$, then not sign-changing ([4, Theorem 1.7]). More generally, for $\mathcal{L}^\mu=-(-\Delta)^{\alpha/2}-\mu$, its solution satisfying $h=O(|x|^{\frac{\alpha-d}{2}})$ is not sign-changing because $\rho=|x|^{\frac{\alpha-d}{2}}\wedge 1$ satisfies (i), (ii) (cf. [20, Theorem 1.2]).

2. Schrödinger forms

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E;m)$. We denote by $u\in\mathcal{D}_{\rm loc}(\mathcal{E})$ if for any relatively compact open set D there exists a function $v\in\mathcal{D}(\mathcal{E})$ such that $u=v$ m-a.e. on D. We denote by $\mathcal{D}_e(\mathcal{E})$ the family of m-measurable functions u on E such that $|u|<\infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty}u_n=u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$.

Let $X=(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\{\mathbf{P}_x\}_{x\in E},\{X_t\}_{t\geq 0},\zeta)$ be the symmetric Hunt process generated by $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the augmented filtration and ζ is the lifetime of X. Denote by $\{p_t\}_{t\geq 0}$ and $\{R_\alpha\}_{\alpha\geq 0}$ the semi-group and resolvent of X:

$$p_tf(x)=\mathbf{E}_x(f(X_t)),\qquad R_\alpha f(x)=\int_0^\infty e^{-\alpha t}p_tf(x)dt.$$

We assume that X satisfies next two conditions:

Irreducibility (I). If a Borel set A is $\{p_t\}_{t\geq 0}$-invariant, that is, $p_t(1_Af)(x)=1_Ap_tf(x)$ m-a.e. for any $f\in L^2(E;m)\cap b\mathscr{B}(E)$ and $t>0$, then A satisfies either $m(A)=0$ or $m(E\setminus A)=0$. Here $b\mathscr{B}(E)$ is the space of bounded Borel functions on E.

Strong Feller Property (SF). For each $t>0$, $p_t(b\mathscr{B}(E))\subset bC(E)$, where $bC(E)$ is the space of bounded continuous functions on E.

We remark that **(SF)** implies the following condition ([11, Theorem 4.2.4]).

Absolute Continuity Condition (AC). The transition probability of E is absolutely continuous with respect to m, $p(t,x,dy) = p(t,x,y)m(dy)$ for each $t > 0$ and $x \in E$.

Under **(AC)**, there exists a non-negative, jointly measurable α-resolvent kernel $r_\alpha(x,y)$:

$$R_\alpha f(x) = \int_E r_\alpha(x,y) f(y) m(dy),\ x \in E,\ f \in b\mathscr{B}(E).$$

Moreover, $r_\alpha(x,y)$ is α-excessive in x and in y ([11, Lemma 4.2.4]). We simply write $r(x,y)$ for $r_0(x,y)$. For a measure μ, we define the α-potential of μ by

$$R_\alpha \mu(x) = \int_E r_\alpha(x,y)\mu(dy).$$

We write $R\mu$ for $R_0\mu$.

We define the *capacity* Cap associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as follows: For an open set $O \subset E$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u) \mid u \in \mathcal{D}(\mathcal{E}), u \geq 1\ m\text{-a.e. on } O\}$$

and for a Borel set $A \subset E$,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) \mid O \text{ is open}, O \supset A\}.$$

where $\mathcal{E}_\alpha(u,u) = \mathcal{E}(u,u) + \alpha(u,u)_m$. A statement depending on $x \in E$ is said to hold *quasi-everywhere* on E if there exists a set $N \subset E$ of zero capacity such that the statement is true for every $x \in E \setminus N$. The notation "q.e." is an abbreviation of quasi-everywhere. A real valued function u defined q.e. on E is said to be *quasi-continuous* if for any $\epsilon > 0$ there exists an open set $G \subset E$ such that $\mathrm{Cap}(G) < \epsilon$ and $u|_{E\setminus G}$ is finite and continuous. Here, $u|_{E\setminus G}$ denotes the restriction of u to $E \setminus G$. Each function u in $\mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$, that is, $u = \tilde{u}$ m-a.e. In the sequel, we always assume that every function $u \in \mathcal{D}_e(\mathcal{E})$ is represented by its quasi-continuous version.

An increasing sequence $\{F_n\}_{n=1}^\infty$ of compact sets is called a *generalized compact nest* if

$$\lim_{n\to\infty} \mathrm{Cap}(K \setminus F_n) = 0 \quad \text{for any compact set } K. \tag{2.1}$$

We call a Borel measure μ on E *smooth* if there exists a generalized compact nest $\{K_n\}_{n=1}^\infty$ such that $\mu(E \setminus \cup_{n=1}^\infty K_n) = 0$ and $1_{K_n} \cdot \mu \in \mathcal{S}_{00}$ for each n, where $\mathcal{S}_{00}$ is the set of finite, positive Radon measure of finite energy with bounded 1-potential, $\sup_{x\in E} R_1\mu(x) < \infty$ ([11, Theorem 2.2.4]). We denote by $\mathcal{S}$ the set of smooth measures. We call a Borel measure μ on E *smooth in the strict sense* if there exists a sequence $\{E_n\}$ of Borel sets such that for each n, $1_{E_n} \cdot \mu \in \mathcal{S}_{00}$ and

$$\mathbf{P}_x(\lim_{n\to\infty} \sigma_n \geq \zeta) = 1, \quad \forall x \in E,$$

where $\sigma_n = \inf\{t > 0 \mid X_t \in E \setminus E_n\}$. In particular, a Radon measure μ with $\sup_{x\in E} R_1\mu(x) < \infty$ is smooth in the strict sense. We denote by $\mathcal{S}_1$ the set of smooth measures in the strict sense.

A stochastic process $\{A_t\}_{t\geq 0}$ is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$.

(ii) there exists a set $\Lambda \in \mathscr{F}_\infty = \sigma\left(\cup_{t\geq 0}\mathscr{F}_t\right)$ and an exceptional set $N \subset E$ such that $\mathbf{P}_x(\Lambda) = 1$, for all $x \in E \setminus N$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying: $A_0 = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation). The set of all PCAF's is denoted by $\mathbf{A}_c^+$. In [11], a PCAF in $\mathbf{A}_c^+$ is called a *PCAF in the strict sense* if $N = \emptyset$. The set of all PCAF's in the strict sense is denoted by $\mathbf{A}_{c,1}^+$. The family $\mathcal{S}$ and $\mathcal{S}_1$ are in one-to-one correspondence to $\mathbf{A}_c^+$ and $\mathbf{A}_{c,1}^+$ respectively (*Revuz correspondence*) as follows: For each $\mu \in \mathcal{S}$ (resp. $\mu \in \mathcal{S}_1$), there exists a unique $\{A_t\}_{t\geq 0} \in \mathbf{A}_c^+$ (resp. $\{A_t\}_{t\geq 0} \in \mathbf{A}_{c,1}^+$) such that for any $f \in \mathscr{B}^+(E)$ and γ-excessive function h ($\gamma \geq 0$), that is, $e^{-\gamma t}p_t h \leq h$,

$$\lim_{t\to 0} \frac{1}{t}\mathbf{E}_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_E f(x)h(x)\mu(dx) \tag{2.2}$$

([11, Theorem 5.1.7]). Here, $\mathbf{E}_{h\cdot m}(\,\cdot\,) = \int_E \mathbf{E}_x(\,\cdot\,)h(x)m(dx)$. We denote by A_t^μ the PCAF corresponding to $\mu \in S$. For a signed smooth measure $\mu = \mu^+ - \mu^-$, we define $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$.

We define some classes of smooth measures.

Definition 2.1. Suppose that $\mu \in \mathcal{S}$.

(1) μ is said to be in the *Kato class* of X ($\mathcal{K}$ in abbreviation) if

$$\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha\mu(x) = 0.$$

μ is said to be in the *local Kato class* ($\mathcal{K}_{\rm loc}$ in abbreviation) if for any compact set K, $1_K \cdot \mu$ belongs to $\mathcal{K}$.

(2) Suppose that X is transient. A measure μ is said to be in the class $\mathcal{K}_\infty$ if for any $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$

$$\sup_{x\in E} R(1_{K^c}\mu)(x) < \epsilon.$$

μ in $\mathcal{K}_\infty$ is called *Green-tight*.

We note that every measure in $\mathcal{K}_{\rm loc}$ is a Radon measure on account of the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the Stollmann-Voigt inequality ([25]): For $\mu \in \mathcal{K}$

$$\int_E u^2 d\mu \leq \|R_\alpha\mu\|_\infty \cdot \mathcal{E}_\alpha(u), \quad \forall u \in \mathcal{D}(\mathcal{E}). \tag{2.3}$$

As a result, $\mathcal{K}_{\rm loc}$ is contained in S_1. We see from [1, Theorem 3.9] that $\mu \in \mathcal{K}$ if and only if

$$\lim_{t\downarrow 0} \sup_{x\in E} \mathbf{E}_x(A_t^\mu) = \lim_{t\downarrow 0} \sup_{x\in E} \int_0^t \int_E p(s,x,y)\mu(dy)ds = 0. \tag{2.4}$$

We denote the Green-tight class by $\mathcal{K}_\infty(X)$ if we would like to emphasize the dependence of the Markov process. Chen [6] defines the Green-tight class in slightly different way; however the two definitions are equivalent under (SF) ([13, Lemma 4.1]).

3. p_t^μ-EXCESSIVE FUNCTIONS AND h-TRANSFORMS

Let $\mu^+ \in \mathcal{K}_{\rm loc}$ and $X^+ = (\mathbf{P}_x^+, X_t, \zeta)$ the subprocess by the multiplicative functional $\exp(-A_t^{\mu^+})$:

$$d\mathbf{P}_x^+ := e^{-A_t^{\mu^+}} d\mathbf{P}_x.$$

In the sequel, we suppose X^+ is irreducible and strong Feller and μ^- is Green-tight with respect to X^+ ($\mu^- \in \mathcal{K}_\infty(X^+)$) to employ Theorem 3.4 below, the analytic criteria for the criticality. For some condition for X^+ being strong Feller, refer [8].

Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_{loc} - \mathcal{K}_\infty(X^+)$. We define the Schrödinger form by

$$\begin{cases} \mathcal{E}^\mu(u) = \mathcal{E}(u) + \displaystyle\int_E u^2 d\mu \\ \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}) \cap L^2(E;\mu^+). \end{cases} \tag{3.1}$$

We see from (2.3) that for a relatively compact open set $D \subset E$,

$$\int_E u^2 d\mu^+ \le \|R_1(1_D\mu^+)\|_\infty \cdot \mathcal{E}_1(u),\ u \in \mathcal{D}(\mathcal{E}) \cap C_0(D)$$

because $1_D\mu^+ \in \mathcal{K}$. Moreover, $\mathcal{K}_\infty(X^+)$ is R^+-*Green bounded*, that is,

$$\sup_{x\in E} R_0^+\mu^-(x) = \sup_{x\in E} \mathbf{E}_x^+[A_\zeta^{\mu^-}] < \infty. \tag{3.2}$$

([6, Proposition 2.2]), and thus

$$\int_E u^2 d\mu^- \le \|R_0^+\mu^-\|_\infty \cdot \mathcal{E}^{\mu^+}(u),\ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Hence we have the next lemma.

Lemma 3.1. *For $\mu = \mu^+ - \mu^- \in \mathcal{K}_{loc} - \mathcal{K}_\infty(X^+)$ and a relatively compact open set $D \subset E$, there exists a positive constant C such that*

$$\left|\int_E u^2 d\mu\right| \le \int_E u^2 d|\mu| \le C \cdot \mathcal{E}(u), \quad u \in \mathcal{D}(\mathcal{E}) \cap C_0(D).$$

Let

$$p_t^\mu h(x) = \mathbf{E}_x\left(e^{-A_t^\mu} h(X_t)\right), \tag{3.3}$$

where $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ is the CAF corresponding to μ. We introduce the function space of p_t^μ-excessive functions:

$$\mathcal{H}_+^{\rm exc}(\mu) = \{h \mid h \text{ is quasi-continuous},\ h > 0 \text{ q.e.},\ p_t^\mu h \le h \text{ q.e.}\}. \tag{3.4}$$

For $h \in \mathcal{H}_+^{\rm exc}(\mu)$ define the bilinear form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ on $L^2(E;h^2m)$ by

$$\begin{cases} \mathcal{E}^{\mu,h}(u) = \mathcal{E}^\mu(hu) \\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(E;h^2m) : hu \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{cases} \tag{3.5}$$

Lemma 3.2. ([16, Proposition 3.4]) *$(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is a quasi-regular Dirichlet form.*

We denote by $\Lambda(\mu)$ the bottom of spectrum of Schrödinger form:

$$\Lambda(\mu) = \inf\left\{ \mathcal{E}^\mu(u) \,\Big|\, \int_E u^2 dm = 1 \right\}. \tag{3.6}$$

If $\Lambda(\mu) \geq 0$, $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is called *positive semi-definite*. It follows from Lemma 3.2 that for $h \in \mathcal{H}_+^{\rm exc}(\mu)$

$$\mathcal{E}^\mu(u) = \mathcal{E}^{\mu,h}(u/h) \geq 0,$$

that is, if $\mathcal{H}_+^{\rm exc}(\mu)$ is not empty, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite.

When $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite, the *extended Schrödinger space* of it can be defined in two ways: First, let $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ be the extended Dirichlet space of the Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is defined in the same manner as in the first paragraph in this section. We define

$$\begin{cases} \mathcal{E}^\mu(u,v) = \mathcal{E}^{\mu,h}\left(\dfrac{u}{h}, \dfrac{v}{h}\right), \\ \mathcal{D}_e(\mathcal{E}^\mu) = \left\{ u : \dfrac{u}{h} \in \mathcal{D}_e(\mathcal{E}^{\mu,h}) \right\}. \end{cases}$$

We established in [27] the *criticality theory* for Schrödinger forms through *h-transform*.

Second, we give, following Schumuland [23], another definition of the extended Schrödinger space in the way similar to extended Dirichlet spaces, the family of m-measurable functions u such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. Denote by $\mathcal{D}_e(\widetilde{\mathcal{E}}^\mu)$ this family, and for $u \in \mathcal{D}_e(\widetilde{\mathcal{E}}^\mu)$ and the sequence $\{u_n\}$ is said to be an *approximating sequence* of u. For $u, v \in \mathcal{D}_e(\widetilde{\mathcal{E}}^\mu)$ and their approximating sequences $\{u_n\}$ and $\{v_n\}$ define

$$\widetilde{\mathcal{E}}^\mu(u,v) = \lim_{n\to\infty} \mathcal{E}^\mu(u_n, v_n). \tag{3.7}$$

Then we see from [23] that $(\widetilde{\mathcal{E}}^\mu, \widetilde{\mathcal{D}_e}(\widetilde{\mathcal{E}}^\mu))$ is well-defined, and have the next lemma.

Lemma 3.3. ([27, Lemma 2.8]) $\mathcal{D}_e(\widetilde{\mathcal{E}}^\mu) = \mathcal{D}_e(\mathcal{E}^\mu)$, $\quad \widetilde{\mathcal{E}}^\mu = \mathcal{E}^\mu$.

We see from Lemma 3.3 that the definition of $\mathcal{D}_e(\mathcal{E}^\mu)$ is independent of $h \in \mathcal{H}_+^{\rm exc}(\mu)$.

The recurrence of a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is characterized in terms of its extended Dirichlet form. Indeed, a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent if the identity function 1 belongs to $\mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(1) = 0$ ([11, Theorem 1.6.3]). As remarked above, if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite, then it is subcritical or critical and the criticality is characterized in terms of its extended Schrödinger form: $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical if and only if there exists a positive function $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ such that $\mathcal{E}^\mu(h) = 0$ and $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical if and only if there exists no positive function $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ such that $\mathcal{E}^\mu(h) = 0$. On account of this observation, we realize that the criticality is an extended concept of the recurrence.

Define

$$\lambda(\mu) = \inf\left\{ \mathcal{E}^{\mu^+}(u) \,\Big|\, u \in \mathcal{D}(\mathcal{E}^\mu),\ \int_E u^2 d\mu^- = 1 \right\}. \tag{3.8}$$

We see that $\lambda(\mu)$ is the principal eigenvalue of the time-changed process of X^+ by PCAF $A_t^{\mu^-}$. We then have an analytic characterization of the criticality in terms of $\lambda(\mu)$:

Theorem 3.4. ([27, Theorem 5.19]) *Let $\mu \in \mathcal{K}_{loc} - \mathcal{K}_\infty(X^+)$.*

(i) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical if and only if $\lambda(\mu) > 1$;*
(ii) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical if and only if $\lambda(\mu) = 1$;*
(iii) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is supercritical if and only if $\lambda(\mu) < 1$.*

Remark 3.5. To show that if $\lambda(\mu) \geq 1$, then $\mathcal{H}_+^{\rm exc}(\mu) \neq \emptyset$, we use the fact that $\mathcal{D}_e(\mathcal{E}^{\mu^+})$ is compactly embedded into $L^2(E;\mu^-)$ under assumption $\mu^- \in \mathcal{K}_\infty(X^+)$ ([29, Theorem 4.8]). Indeed, when $\lambda(\mu) = 1$ there exists $h \in \mathcal{D}_e(\mathcal{E}^{\mu^+})(\subset \mathcal{D}_e(\mathcal{E}^\mu))$ such that $p_t^\mu h = h$ (see [27, 5.2]). When $\lambda(\mu) > 1$ define $\mu' = \mu^+ - \lambda(\mu)\mu^-$. Then

$$\lambda(\mu') = \inf\left\{\mathcal{E}(u) + \int_E u^2 d\mu^+ \,\middle|\, \lambda(\mu)\int_E u^2 d\mu^- = 1\right\} = 1.$$

Denoting by h' the minimizer of $\lambda(\mu')$, we see that $p_t^\mu h' \leq p_t^{\mu'} h' = h'$ and $h' \in \mathcal{H}_+^{\rm exc}(\mu)$. We will give in Section 5 another function in $\mathcal{H}_+^{\rm exc}(\mu)$ defined as a Feynman-Kac functional, so called, gauge function.

Lemma 3.6. (i) $\Lambda(\mu) < 0 \Longleftrightarrow \lambda(\mu) < 1$;
(ii) $\Lambda(\mu) > 0 \Longrightarrow \lambda(\mu) > 1$;
(iii) $\lambda(\mu) = 1 \Longrightarrow \Lambda(\mu) = 0$.

Proof. Statements (i) and (ii) are shown in [31, Lemma 2.2] and [28, Lemma 3.5] respectively. Since the criticality of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ and $\lambda(\mu) = 1$ are equivalent, statements (iii) follows from (i) and (ii). □

Lemma 3.6 says that for $\mu \in \mathcal{K}_{loc} - \mathcal{K}_\infty(X^+)$ the subcriticality (resp. criticality) cannot be characterized by $\Lambda(\mu) > 0$ (resp. $\Lambda(\mu) = 0$). If $\Lambda(\mu^+) > 0$, then $\Lambda(\mu) > 0$ is equivalent to $\lambda(\mu) > 1$, consequently $\Lambda(\mu) = 0$ is equivalent to $\lambda(\mu) = 1$ ([28, Lemma 3.5]).

Remark 3.7. Define

$$\widetilde{\lambda}(\mu) = \inf\left\{\mathcal{E}^\mu(u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}^\mu), \int_E u^2 d\mu^- = 1\right\}.$$

Then $\widetilde{\lambda}(\mu) = \lambda(\mu) - 1$ and Lemma 3.5 is rewritten as (i) $\Lambda(\mu) < 0 \Longleftrightarrow \widetilde{\lambda}(\mu) < 0$, (ii) $\Lambda(\mu) > 0 \Longrightarrow \widetilde{\lambda}(\mu) > 0$, (iii) $\widetilde{\lambda}(\mu) = 0 \Longrightarrow \Lambda(\mu) = 0$.

Lemma 3.8. ([30, Lemma 3.9]) *If $\lambda(\mu) > 1$ and $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ satisfies $\mathcal{E}^\mu(h) = 0$, then $h \equiv 0$.*

Proof. For $g \in \mathcal{H}_+^{\rm exc}(\mu)$, $(\mathcal{E}^{\mu,g}, \mathcal{D}(\mathcal{E}^{\mu,g}))$ is transient and so $(\mathcal{D}_e(\mathcal{E}^{\mu,g}), \mathcal{E}^{\mu,g})$ is a Hilbert space. Since $h/g \in \mathcal{D}_e(\mathcal{E}^{\mu,g})$,

$$\mathcal{E}^{\mu,g}(h/g) = \mathcal{E}^\mu(h) = 0,$$

which implies $h/g \equiv 0$ and so $h \equiv 0$. □

We see from Lemma 3.8 that if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, then there is no non-trivial solution in $\mathcal{D}_e(\mathcal{E}^\mu)$.

Lemma 3.9. *If $\lambda(\mu) = 1$, then there exists $h \in \mathcal{H}_+^{exc}(\mu)$ belonging to $\mathcal{D}_e(\mathcal{E}^\mu) \cap bC(E)$ and it satisfies*

$$\mathcal{E}^\mu(h, \varphi) = 0 \quad \textit{and} \quad \mathcal{E}^\mu(h\varphi) = \mathcal{E}^h(\varphi) \ \forall \varphi \in \mathcal{D}(\mathcal{E}). \tag{3.9}$$

Proof. As noted in Remark 3.5, there exists $h \in \mathcal{D}_e(\mathcal{E}^{\mu^+}) (\subset \mathcal{D}_e(\mathcal{E}^\mu))$ such that $\mathcal{E}^\mu(h) = 0$. Since $\Lambda(\mu) = 0$, h is the minimizer in (3.6), which implies (3.9). □

Lemma 3.10. ([30, Lemma 3.10]) *Suppose $\lambda(\mu) = 1$. If $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ satisfies $\mathcal{E}^\mu(h) = 0$, then h is written as $h = cg$, $c \in \mathbb{R}$, $g \in \mathcal{H}_+^{\rm exc}(\mu)$.*

Proof. Let $g \in \mathcal{H}_+^{\rm exc}(\mu)$. Since $h/g \in \mathcal{D}_e(\mathcal{E}^{\mu,g})$ and $\mathcal{E}^{\mu,g}(h/g) = 0$, $h/g \equiv c$ by the recurrence of $(\mathcal{E}^{\mu,g}, \mathcal{D}(\mathcal{E}^{\mu,g}))$. □

Combining Lemma 3.8 with Lemma 3.10, we have the next corollary.

Corollary 3.11. *Suppose $\Lambda(\mu) \geq 0$. Then if $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ satisfies $\mathcal{E}^\mu(h) = 0$, then h is not sign-changing.*

4. $\mathcal{L}^\mu$-SUPERSOLUTIONS AND STRONG MAXIMUM PRINCIPLE

We denote by $\mathcal{L}^\mu = \mathcal{L} - \mu$ the self-adjoint operator associated with the closed symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, $(-\mathcal{L}^\mu u, v)_m = \mathcal{E}^\mu(u, v)$. A function $h \in \mathcal{D}_{\rm loc}^\dagger(\mathcal{E}) \cap L_{\rm loc}^\infty(E; m)$ is called a *solution* (*subsolution*, *supersolution*) to $\mathcal{L}^\mu u = 0$ if

$$\mathcal{E}^\mu(h, \varphi) = 0 \ (\leq 0, \ \geq 0) \text{ for } \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E).$$

Denote by $\mathcal{H}_+^{\rm sol}(\mu)$ (resp. $\mathcal{H}_+^{\rm sub}(\mu)$, $\mathcal{H}_+^{\rm sup}(\mu)$) the function space of positive solutions (resp. subsolution, supersolution). Following the argument in the proof of [21, Theorem 5.1], we have the next lemma.

Lemma 4.1. *It holds that*

$$\mathcal{H}_+^{exc}(\mu) \cap \mathcal{D}_{loc}^\dagger(\mathcal{E}) \cap L_{loc}^\infty(E; m) \subset \mathcal{H}_+^{sup}(\mu).$$

Proof. Let $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E)$ and K the support of φ. Let G be a relatively compact open set such that $K \subset G$ and $\{\psi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0^+(E)$ such that $\psi_n \leq 1$ on E, $\psi_n = 1$ on G, and $\psi_n(x) \uparrow 1$ as $n \to \infty$. Then for $h \in \mathcal{H}_+^{\rm exc}(\mu) \cap \mathcal{D}_{\rm loc}^\dagger(\mathcal{E}) \cap L_{\rm loc}^\infty(E; m)$

$$\mathcal{E}(h\psi_n, \varphi) + \int_E h\psi_n \varphi \, d\mu \geq 0.$$

Indeed, the left-hand side is equal to

$$\lim_{t \downarrow 0} \frac{1}{t} \big(h\psi_n - p_t^\mu(h\psi_n), \varphi\big)_m = \lim_{t \downarrow 0} \frac{1}{t} \Big(\big(h, \varphi\big)_m - \big(p_t^\mu(h\psi_n), \varphi\big)_m \Big).$$

This limit is nonnegative because $p_t^\mu(h\psi_n) \le p_t^\mu h \le h$. Since $h\psi_n = h$ q.e. on G, $\mathcal{E}(h\psi_n, \varphi)$ is equal to

$$\frac{1}{2}\int_E d\mu^c_{\langle h,\varphi\rangle} + \iint_{K\times K}(h(x)-h(y))(\varphi(x)-\varphi(y))J(dx,dy)$$
$$+ 2\iint_{K\times(K^c\cap G)}(h(x)-h(y))(\varphi(x)-\varphi(y))J(dx,dy)$$
$$+ 2\iint_{K\times G^c}(h(x)-h(y)\psi_n(y))\cdot\varphi(x)\,J(dx,dy) + \int_E h\varphi\,d\kappa.$$

Noting $J(K\times G^c) < \infty$ and $h \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E}) \cap L^\infty_{\mathrm{loc}}(E;m)$, we see

$$\begin{aligned}|h(x)-h(y)\psi_n(y)\varphi(x)| &\le |h(x)-h(x)\psi_n(y)|\varphi(x) + |h(x)-h(y)||\psi_n(y)\varphi(x)| \\ &\le |h(x)|\varphi(x)1_{G^c}(y) + |h(x)-h(y)|\varphi(x) \in L^1(K\times G^c;J).\end{aligned}$$

From the dominated convergence theorem the fourth term tends to

$$2\iint_{K\times G^c}(h(x)-h(y))\cdot\varphi(x)\,J(dx,dy)$$

as $n\to\infty$ and $\lim_{n\to\infty}\mathcal{E}(h\psi_n,\varphi) = \mathcal{E}(h,\varphi)$. Hence

$$\mathcal{E}(h,\varphi) + \int_E h\varphi\,d\mu = \lim_{n\to\infty}\left(\mathcal{E}(h\psi_n,\varphi) + \int_E h\psi_n\varphi\,d\mu\right) \ge 0.$$

□

As an application of the generalized Fukushima decomposition for $\mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ (Kuwae [14], [15]), we prove the strong maximum principle.

Lemma 4.2. *For a non-negative supersolution h, there exists a smooth Radon measure ν_h such that*

$$\mathcal{E}^\mu(h,\varphi) = \int_E \varphi d\nu_h, \quad \varphi \in \mathcal{D}(\mathcal{E})\cap C_0(E).$$

Proof. Denote $\mathcal{L} = \mathcal{D}(\mathcal{E})\cap C_0(E)$ and define a function I on $\mathcal{L}$ by

$$I(\varphi) = \mathcal{E}^\mu(h,\varphi), \quad \varphi\in\mathcal{L}. \tag{4.1}$$

We confirm in [27, Lemma 4.7] that $\mathcal{L}$ is a *Stone vector lattice* and $I(\varphi)$ is a *pre-integral* (For these definitions, see [9, p.143]). Hence, we know from Stone-Daniell theorem [9, Theorem 4.5.2] that there exists a Borel measure ν such that

$$I(\varphi) = \int_E \varphi d\nu_h, \quad \forall\varphi\in\mathcal{L}.$$

Moreover, we can show by the same argument as in [27, Lemma 4.7] and [21, Lemma 4.1] that the measure ν_h is smooth. Indeed, let K be a compact set with $\mathrm{Cap}(K) = 0$. Take relatively compact open sets G and D such that $K\subset G\subset\bar{G}\subset D\subset E$. Let $\{\varphi_n\}$ be a subset of $\mathcal{D}(\mathcal{E})\cap C_0^+(G)$ such that $\varphi_n \ge 1$ on K, and $\lim_{n\to\infty}\mathcal{E}(\varphi_n) = 0$. The existence of such a sequence $\{\varphi_n\}$ follows from [11, Lemma 2.2.7, Theorem 4.4.3]. Let $\psi\in\mathcal{D}(\mathcal{E})\cap C_0^+(D)$ such that $\psi \le 1$ and $\psi = 1$ on G. Then, we can show $\mathcal{E}^\mu(h,\varphi_n) \le \mathcal{E}^\mu(h\psi,\varphi_n)$ by the argument in the proof of [21, Lemma 4.1]. Indeed,

$$\begin{aligned}&\iint_{E\times E}(h(x)-h(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy)\\ =&\iint_{G\times G}(h(x)-h(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy)\\ +2&\iint_{G\times G^c}(h(x)-h(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy).\end{aligned}$$

The first term of the right hand side equals

$$\iint_{G\times G}(h(x)\psi(x)-h(y)\psi(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy).$$

and the second term equals

$$2\iint_{G\times G^c}(h(x)-h(y))\varphi_n(x)J(dx,dy).$$

Since

$$h(x)-h(y)=h(x)\psi(x)-h(y)\le h(x)\psi(x)-h(y)\psi(y),\ (x,y)\in G\times G^c,$$

the second term is less than or equal to

$$2\iint_{G\times G^c}(h(x)\psi(x)-h(y)\psi(y))(\varphi_n(x)-\varphi_n(y))J(dx,dy),$$

which leads us to $\mathcal{E}^\mu(h,\varphi_n) \le \mathcal{E}^\mu(h\psi,\varphi_n)$. Hence we have

$$\nu_h(K)\le\int_E\varphi_n d\nu_h=\mathcal{E}^\mu(h,\varphi_n)\le\mathcal{E}^\mu(h\psi,\varphi_n). \tag{4.2}$$

Since

$$\begin{aligned}\left|\int_E h\psi\varphi_n d\mu\right| &\le \left(\int_E (h\psi)^2 d|\mu|\right)^{1/2}\cdot\left(\int_E \varphi_n^2 d|\mu|\right)^{1/2}\\ &\le C\left(\int_E (h\psi)^2 d|\mu|\right)^{1/2}\cdot\mathcal{E}_1(\varphi_n)^{1/2}\longrightarrow 0\ \text{ as } n\to\infty\end{aligned}$$

by Lemma 3.1, $\lim_{n\to\infty}\mathcal{E}^\mu(h\psi,\varphi_n)=0$ and $\nu_h(K)=0$ by (4.2). The measure ν_h is Radon by (4.2) and so smooth. □

The next lemma is an extension of [30, Lemma 3.18, Lemma 3.19], where a function in $\mathcal{D}_e(\mathcal{E})$ is treated.

Lemma 4.3. *If h is a non-negative supersolution, then $p_t^\mu h(x)\le h(x)$, q.e. x.*

Proof. Denote $\eta = k+\mu$, where k is the killing measure in the Beuring-Deny formula of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. Let $(\mathcal{E}^{\rm res},\mathcal{D}(\mathcal{E}^{\rm res}))$ be the *resurrected Dirichlet form* defined in [7, (5.2.25)], which is a regular Dirichlet form on $L^2(E;m)$. We see from [7, Theorem 5.2.17] that $\mathcal{D}(\mathcal{E}^{\rm res})\supset\mathcal{D}(\mathcal{E})$ and $\mathcal{D}(\mathcal{E}^{\rm res})\cap L^2(E;k)=\mathcal{D}(\mathcal{E})$.

As a result, we know that $\mathcal{D}(\mathcal{E}^{\rm res})\cap C_0(E)=\mathcal{D}(\mathcal{E})\cap C_0(E)$ and $\mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E}^{\rm res})\supset\mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E})$ and that

$$\mathcal{E}^{\rm res}(h,\varphi)=-\int_X\varphi(h\cdot d\eta-d\nu_h),\quad \varphi\in\mathcal{D}(\mathcal{E}^{\rm res})\cap C_0(E).$$

Since η is a smooth Radon measure, so is $h\cdot\eta$ for $h\in\mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E})\cap L^1_{\rm loc}(E;|\mu|)$.

Let $X^{\rm res}=(\mathbf{P}^{\rm res}_x,X_t)$ be the *resurrected process* of X, that is, the Hunt process generated by $(\mathcal{E}^{\rm res},\mathcal{D}(\mathcal{E}^{\rm res}))$ (cf. [7, Theorem 5.2.17]). We then see from [21, Corollary 3.3] that

$$h(X_t)=h(X_0)+M^{[h]}_t+\int_0^t h(X_s)dA^{\eta}_s-A^{\nu_h}_t,\quad t<\zeta^{\rm res},\quad \mathbf{P}^{\rm res}_x\text{-a.s. q.e. } x,$$

where $M^{[h]}_t\in\mathcal{M}^{[[0,\zeta^{\rm res}[[}_{\rm loc}$ is the martingale part of the Fukushima decomposition, i.e., there exist a sequence $\{S_n\}$ of stopping times and a sequence $\{M^n_t\}$ of square integrable martingale AFs such that $S_n\uparrow\zeta^{\rm res}$ and

$$M^{[h]}_{t\wedge S_n}1_{\{t\wedge S_n<\zeta^{\rm res}\}}=M^n_{t\wedge S_n}1_{\{t\wedge S_n<\zeta^{\rm res}\}}$$

([14, Theorem 4.2]).

Let $h_\varepsilon=h\vee\varepsilon$, $\varepsilon>0$ and $\nu_\varepsilon=(1/h_\varepsilon)\nu_h$. Define $T_n=\inf\{t>0\mid A^{\eta}_t+A^{\nu_\varepsilon}_t>n\}(n=1,2,\ldots)$. Note that $\mathbf{P}^{\rm res}_x(X_{\zeta-}=X_\zeta,\zeta<\infty)=0$ because $(\mathcal{E}^{\rm res},\mathcal{D}(\mathcal{E}^{\rm res}))$ has no killing part ([11, Theorem 5.3.1]). Consequently, $\zeta^{\rm res}$ is a predictable stopping time, i.e., there exists a sequence $\{\zeta_n\}$ of stopping times such that $\zeta_n<\zeta^{\rm res}$ and $\zeta_n\uparrow\zeta^{\rm res}$. Define $\sigma_n=S_n\wedge T_n\wedge\zeta_n$. Then $\sigma_n<\zeta^{\rm res}$ and $\sigma_n\uparrow\zeta^{\rm res}$. Put $C_t=A^{\eta}_t-A^{\nu_\varepsilon}_t$. We then see from Itô's formula that

$$\begin{aligned}e^{-C_{t\wedge\sigma_n}}h(X_{t\wedge\sigma_n})&=h(X_0)+\int_0^{t\wedge\sigma_n}e^{-C_s}h(X_s)(-dC_s)+\int_0^{t\wedge\sigma_n}e^{-C_s}dM^{[h]}_s\\&\quad+\int_0^{t\wedge\sigma_n}e^{-C_s}(h(X_s)dA^{\eta}_s-dA^{\nu_h}_s)\\&=h(X_0)+\int_0^{t\wedge\sigma_n}e^{-C_s}dM^{[h]}_s+\int_0^{t\wedge\sigma_n}e^{-C_s}h(X_s)dA^{\nu_\varepsilon}_s\\&\quad-\int_0^{t\wedge\sigma_n}e^{-C_s}dA^{\nu_h}_s,\quad \mathbf{P}^{\rm res}_x\text{-a.s. q.e. } x.\end{aligned}$$

Noting that $\int_0^{t\wedge\sigma_n}e^{-C_s}dM^{[h]}_s$ is a martingale and

$$h(X_s)dA^{\nu_\varepsilon}_s=\frac{h(X_s)}{h_\varepsilon(X_s)}dA^{\nu_h}_s\le dA^{\nu_h}_s,$$

we have

$$\mathbf{E}^{\rm res}_x\left(e^{-A^{\eta}_{t\wedge\sigma_n}}h(X_{t\wedge\sigma_n})\right)\le\mathbf{E}^{\rm res}_x\left(e^{-C_{t\wedge\sigma_n}}h(X_{t\wedge\sigma_n})\right)\le h(x),\quad \text{q.e. } x.$$

By Fatou's lemma

$$\begin{aligned}\mathbf{E}^{\rm res}_x\left(e^{-A^{\eta}_t}h(X_t)\right)&=\mathbf{E}^{\rm res}_x\left(e^{-A^{\eta}_{t\wedge\zeta^{\rm res}}}h(X_{t\wedge\zeta^{\rm res}})\right)\\&\le\liminf_{n\to\infty}\mathbf{E}^{\rm res}_x\left(e^{-A^{\eta}_{t\wedge\sigma_n}}h(X_{t\wedge\sigma_n})\right)\le h(x),\quad \text{q.e. } x.\end{aligned}\tag{4.3}$$

By applying [7, Theorem 5.2.17] again, we see that the left hand side of (4.3) is equal to $\mathbf{E}_x\left(\exp\left(-A_t^{\mu}\right) h(X_t)\right) = p_t^{\mu} h(x)$. □

Let h be a non-negative supersolution and put $S = \{x \in E \mid h(x) > 0\}$. If $\mathrm{Cap}(S) > 0$, then

$$\mathbf{P}_x(\sigma_S < \zeta) > 0, \quad \text{q.e. } x$$

by the irreducibility of X ([11, Theorem 4.7.1]). Moreover, by the quasi-continuity of h, the set S is a quasi open and thus q.e. finely open by [11, Theorem 4.6.1]. Hence,

$$\mathbf{P}_x\left(\int_0^{\zeta} 1_S(X_t)dt > 0\right) > 0, \quad \text{q.e. } x,$$

and thus

$$0 < \alpha \int_0^{\infty} e^{-\alpha t} p_t^{\mu} h(x) dt \leq \alpha \int_0^{\infty} e^{-\alpha t} h(x) dt \leq h(x), \text{ q.e. } x$$

by Lemma 4.3. Therefore, we have the next theorem.

Theorem 4.4. *Suppose X is irreducible. A non-negative supersolution is positive q.e. or equal to zero q.e.*

The property in Theorem 4.4 is called the *strong maximum principle* and considered in [2], [5].

Remark 4.5. Theorem 4.4 can be extended as follows: Let $\mu = \mu^+ - \mu^-$ be a Borel signed measure. Suppose that there exists a exceptional closed set K such that the restriction $\mu^D = \mu^{D,+} - \mu^{D,-}$of μ to $D = E \setminus K$ is in $\mathcal{K}_{\mathrm{loc}}(X^D) - \mathcal{K}_{\infty}(X^{D,+})$, where X^D, $X^{D,+}$ are the part process of X on D and its subprocess by $\exp(-A_t^{\mu^+})$. Let $h \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E}^D) \cap L^{\infty}_{\mathrm{loc}}(E; m)$ such that

$$\mathcal{E}^{\mu}(h, \varphi) \geq 0, \quad \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(D).$$

Then the function h is positive quasi-everywhere on D or equal to zero quasi-everywhere on D, which implies "quasi-everywhere on E" because $\mathrm{Cap}(K) = 0$.

5. Existence of positive bounded solutions

In this section, assume X^+ satisfies (I) and (SF) and $\mu^- \in \mathcal{K}_{\infty}(X^+)$. Let $g^{\mu}(x)$ be a so-called *gauge function* defined by

$$g^{\mu}(x) = \mathbf{E}_x^+\left(e^{A_{\zeta}^{\mu^-}}\right). \tag{5.1}$$

Then it is known in [6, Theorem 5.1] that g^{μ} is a bounded function if and only if $\lambda(\mu) > 1$.

Lemma 5.1. ([27, Lemma 5.2]) *Suppose $\lambda(\mu) > 1$. The function g^{μ} is p_t^{μ}-excessive; $p_t^{\mu} g^{\mu}(x) \uparrow g^{\mu}(x)$ as $t \downarrow 0$.*

Proof. By the Markov property of X

$$\begin{aligned}\mathbf{E}_x\left(e^{-A_t^{\mu}}g^{\mu}(X_t);t<\zeta\right)&=\mathbf{E}_x^+\left(e^{A_t^{\mu^-}}g^{\mu}(X_t);t<\zeta\right)\\&=\mathbf{E}_x^+\left(e^{A_t^{\mu^-}}\mathbf{E}_{X_t}^+\left(e^{A_\zeta^{\mu^-}}\right);t<\zeta\right)\\&=\mathbf{E}_x^+\left(\mathbf{E}_x^+\left(e^{A_t^{\mu^-}+A_\zeta^{\mu^-}(\theta_t)}1_{\{t<\zeta\}}\Big|\mathscr{F}_t\right)\right).\end{aligned}$$

The right-hand side equals $\mathbf{E}_x^+\left(e^{A_\zeta^{\mu^-}};t<\zeta\right)$ because $A_t^{\mu^-}+A_\zeta^{\mu^-}(\theta_t)=A_\zeta^{\mu^-}$ on $\{t<\zeta\}$. Therefore, as $t\downarrow 0$

$$p_t^{\mu}g^{\mu}(x)=\mathbf{E}_x^+\left(e^{A_\zeta^{\mu^-}};t<\zeta\right)\uparrow\mathbf{E}_x^+\left(e^{A_\zeta^{\mu^-}}\right)=g^{\mu}(x).$$

□

Lemma 5.2. *Suppose* $\lambda(\mu)>1$. *Then it holds that*

$$g^{\mu}(x)=1+R^+(g^{\mu}\mu^-)(x). \tag{5.2}$$

Proof. Define a uniformly integrable martingale M_t by

$$M_t=\mathbf{E}_x^+\left(e^{A_\zeta^{\mu^-}}\Big|\mathscr{F}_t\right).$$

Then

$$\int_0^t g^{\mu}(X_s)dA_s^{\mu^-}=\int_0^t e^{-A_s^{\mu^-}}M_s dA_s^{\mu^-},\quad t<\zeta \tag{5.3}$$

because

$$\begin{aligned}e^{-A_t^{\mu^-}}M_t1_{\{t<\zeta\}}&=e^{-A_t^{\mu^-}}\mathbf{E}_x^+\left(e^{A_\zeta^{\mu^-}}1_{\{t<\zeta\}}\Big|\mathscr{F}_t\right)\\&=e^{-A_t^{\mu^-}}\mathbf{E}_x^+\left(e^{A_t^{\mu^-}+A_\zeta^{\mu^-}(\theta_t)}1_{\{t<\zeta\}}\Big|\mathscr{F}_t\right)\\&=\mathbf{E}_{X_t}^+\left(e^{A_\zeta^{\mu^-}}\right)1_{\{t<\zeta\}}=g^{\mu}(X_t)1_{\{t<\zeta\}}.\end{aligned}$$

By Itô's formula,

$$e^{-A_t^{\mu^-}}M_t=M_0+\int_0^t e^{-A_s^{\mu^-}}dM_s-\int_0^t e^{-A_s^{\mu^-}}M_s dA_s^{\mu^-},$$

and thus

$$\mathbf{E}_x^+(M_0)=\mathbf{E}_x^+\left(e^{-A_\zeta^{\mu^-}}M_\zeta\right)+\mathbf{E}_x^+\left(\int_0^\zeta e^{-A_s^{\mu^-}}M_s dA_s^{\mu^-}\right).$$

Noting that $\mathbf{E}_x^+(M_0)=g^{\mu}(x)$, $e^{-A_\zeta^{\mu^-}}M_\zeta=e^{-A_\zeta^{\mu^-}}e^{A_\zeta^{\mu^-}}=1$ and by (5.3)

$$\mathbf{E}_x^+\left(\int_0^\zeta e^{-A_s^{\mu^-}}M_s dA_s^{\mu^-}\right)=\mathbf{E}_x^+\left(\int_0^\zeta g^{\mu}(X_s)dA_s^{\mu^-}\right)=R^+(g^{\mu}\mu^-)(x),$$

we have this lemma. □

Lemma 5.3. ([27, Lemma 5.4]) *If* $\lambda(\mu) > 1$*, then* g^μ *belongs to* $\mathcal{D}^{\dagger}_{loc}(\mathcal{E}) \cap bC(E)$*.*

Proof. On account of Lemma 5.2, we have only to prove that $R^+(g^\mu\mu^-) \in \mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E}) \cap bC(E)$. Noting that $g^\mu\mu^- \in \mathcal{K}_\infty(X^+)$, we have $R^+(g^\mu\mu^-) \in b\mathscr{B}(E)$ by [6, Proposition 2.2] and $R^+_\alpha(R^+(g^\mu\mu^-)) \in bC(E)$ by the resolvent strong Feller property of X^+. Since

$$\|R^+(g^\mu\mu^-) - \alpha R^+_\alpha(R^+(g^\mu\mu^-))\|_\infty = \|R^+_\alpha(g^\mu\mu^-)\|_\infty \to 0, \quad \alpha \to \infty$$

by $g^\mu\mu^- \in \mathcal{K}_\infty(X^+)(\subset \mathcal{K}(X^+))$, $R^+(g^\mu\mu^-)$ also belongs to $bC(E)$. By the same argument as in [13, Theorem 3], we can prove $R^+(g^\mu\mu^-) \in \mathcal{D}_{\rm loc}(\mathcal{E}^+)(\subset \mathcal{D}_{\rm loc}(\mathcal{E}))$. Therefore g^μ belongs to $\mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E}) \cap C(E)$ on account of $b\mathcal{D}_{loc}(\mathcal{E}) \subset \mathcal{D}^{\dagger}_{loc}(\mathcal{E})$. □

We introduce

$$\mathcal{H}^{\rm inv}_+(\mu) = \{h \mid h \text{ is quasi-continuous, } h > 0 \text{ q.e., } p^\mu_t h = h \text{ q.e.}\}.$$

Denote by $\mathcal{L}^\mu_1$ and $\mathcal{L}^\mu_2$ the L^1-generator of p^μ_t and L^2-one respectively. We make the following assumption.

(A). There exists a core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $\mathcal{C} \subset \mathcal{D}(\mathcal{L}^\mu_1) \cap \mathcal{D}(\mathcal{L}^\mu_2) \cap C_0(E)$.

Lemma 5.4. *Under Assumption (A),* $\mathcal{H}^{\rm inv}_+(\mu) \cap b\mathcal{D}^{\dagger}_{loc}(\mathcal{E})$ *is contained in* $\mathcal{H}^{\rm sol}_+(\mu)$*.*

Proof. For $h \in \mathcal{H}^{\rm inv}_+(\mu) \cap b\mathcal{D}^{\dagger}_{\rm loc}(\mathcal{E})$ and $\varphi \in \mathcal{C}$,

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{1}{t}(h - p^\mu_t h, \varphi)_m = \lim_{t\to 0} \frac{1}{t}(h, \varphi - p^\mu_t \varphi)_m \\ &= (h, -\mathcal{L}^\mu_1\varphi)_m = (h, -\mathcal{L}^\mu_2\varphi)_m. \end{aligned}$$

Let $\{\varphi_n\}$ be a sequence of functions in $\mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $0 \le \varphi_n \le 1$, $\varphi_n \uparrow 1$, $\varphi_1 = 1$ on $\mathrm{supp}[\varphi]$. Then

$$(h, -\mathcal{L}^\mu_2\varphi)_m = \lim_{n\to\infty}(h\varphi_n, -\mathcal{L}^\mu_2\varphi)_m = \lim_{n\to\infty}\mathcal{E}^\mu(h\varphi_n, \varphi).$$

By the same argument as in Lemma 4.1, the right hand side equals $\mathcal{E}^\mu(h,\varphi)$. Hence, $\mathcal{E}^\mu(h,\varphi) = 0$ and $h \in \mathcal{H}^{\rm sol}_+(\mu)$. □

Moreover, we assume

(B). $\mathbf{P}^+_x(\zeta = \infty) = \mathbf{E}_x\left(e^{-A^{\mu^+}_\infty}\right) > 0, \quad \forall x \in E.$

Example 5.5. Let $(\mathbf{P}^W_x, B_t)$ be the Brownian motion. Assume $d \ge 3$ and V^+ and V^- are bounded functions on $\mathbb{R}^d$ with $V = V^+ - V^- \in \mathcal{K}_D - \mathcal{K}_\infty$, Here $\mathcal{K}_D$ is the class of bounded potential, i.e.

$$\int_{\mathbb{R}^d} \frac{V^+(y)dy}{|x-y|^{d-2}} < \infty.$$

Let $\mathbf{P}^+_x$ be the subprocess of the Brownian motion defined by

$$d\mathbf{P}^+_x = e^{-\int_0^t V^+(B_s)ds} d\mathbf{P}^W_x.$$

Then the assumptions (A) and (B) are satisfied because

$$\mathbf{P}_x^+(\zeta=\infty)=\mathbf{E}_x^W\left(e^{-\int_0^\infty V^+(B_s)ds}\right)>0.$$

Define

$$\widetilde{g}^V(x)=\mathbf{E}_x^W\left(e^{-\int_0^\infty V(B_t)dt}\right).$$

Then $\widetilde{g}^V$ belongs to $\mathcal{D}_{\rm loc}(\mathcal{E})\cap bC(\mathbb{R}^d)\cap\mathcal{H}_+^{\rm inv}(V)$, as a result, $\widetilde{g}^V$ is a solution to $(1/2)\Delta h-Vh=0$.

6. Liouville-type theorem

We define a positive bilinear form by

$$\mathcal{E}^h(\varphi)=\frac{1}{2}\int_E h^2d\mu^c_{\langle\varphi\rangle}+\iint_{E\times E\setminus d}h(x)h(y)(\varphi(x)-\varphi(y))^2J(dx,dy), \tag{6.1}$$

where $\mu^c_{\langle\varphi\rangle}(=\mu^c_{\langle\varphi,\varphi\rangle})$ is the local part of energy measure of $\mu_{\langle\varphi\rangle}(=\mu_{\langle\varphi,\varphi\rangle})$, $\varphi\in\mathcal{D}(\mathcal{E})$(cf. [11, p. 126]).

Let $h\in\mathcal{D}^\dagger_{loc}(\mathcal{E})\cap L^\infty_{\rm loc}(E;m)$ and $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$ with $\mathrm{supp}[\varphi]=K$. Then $h\varphi$ belongs to $\mathcal{D}(\mathcal{E})$. Indeed, put $l=\sup_{x\in K}|h(x)|$. Since $h^{(l)}=((-l)\vee h)\wedge l\in b\mathcal{D}_{\rm loc}(\mathcal{E})$ and $h^{(l)}\varphi\in\mathcal{D}(\mathcal{E})$, $h\varphi(=h^{(l)}\varphi)$ belongs to $\mathcal{D}(\mathcal{E})$ and $((h\varphi)(x)-(h\varphi)(y))^2\in L^1(E\times E\setminus d;J)$. In addition, the function $(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))$ belongs to $L^1(E\times E\setminus d;J)$ because

$$\begin{aligned}
&\iint_{E\times E\setminus d}|(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))|J(dx,dy)\\
=&\iint_{K\times E\setminus d}|(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))|J(dx,dy)\\
&+\iint_{K^c\times E\setminus d}|(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))|J(dx,dy)\\
=&\iint_{K\times E\setminus d}|(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))|J(dx,dy)\\
&+\iint_{K^c\times K\setminus d}|(h(x)-h(y))((h\varphi^2)(x)-(h\varphi^2)(y))|J(dx,dy)\\
\leq&\, 2\left(\iint_{K\times E\setminus d}(h(x)-h(y))^2J(dx,dy)\right)^{1/2}\\
&\times\left(\iint_{K\times E\setminus d}((h\varphi^2)(x)-(h\varphi^2)(y))^2J(dx,dy)\right)^{1/2}<\infty
\end{aligned}$$

by $h\in\mathcal{D}^\dagger_{loc}(\mathcal{E})$ and $h\varphi^2\in\mathcal{D}(\mathcal{E})$. Hence $h(x)h(y)(\varphi(x)-\varphi(y))^2=(h(x)\varphi(x)-h(y)\varphi(y))^2-(h(x)-h(y))(h(x)\varphi^2(x)-h(y)\varphi^2(y))\in L^1(E\times E\setminus d;J)$. Consequently, if $h\in\mathcal{D}^\dagger_{loc}(\mathcal{E})\cap L^\infty_{\rm loc}(E;m)$, then $\mathcal{E}^h(\varphi)<\infty$ for $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$. Since

$$d\mu^c_{\langle h\varphi\rangle}-h^2d\mu^c_{\langle\varphi\rangle}=\mu^c_{\langle h,h\varphi^2\rangle}$$

by the derivation property of $\mu^c_{\langle u,v\rangle}$ ([11, Lemma 3.25]), we have the next key equation for the proof of Liouville-type theorem: For $h\in\mathcal{D}^\dagger_{loc}(\mathcal{E})\cap L^\infty_{\rm loc}(E;m)$

and $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$

$$\mathcal{E}^\mu(h\varphi) = \mathcal{E}^h(\varphi) + \mathcal{E}^\mu(h, h\varphi^2). \tag{6.2}$$

Theorem 6.1. *Let $\mu \in \mathcal{K}_{loc} - \mathcal{K}_\infty(X^+)$.*
(i) *Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent and $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite. $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical ($\Longleftrightarrow \lambda(\mu) = 1$) if and only if there exists a positive bounded solution h.*
(ii) *Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient and its generating Markov process satisfies (A) and (B). Then $\lambda(\mu) \geq 1 (\Longleftrightarrow \Lambda(\mu) \geq 0)$ if and only if there exists a positive bounded solution to $\mathcal{L}^\mu h = 0$.*

Proof. (i) ($\Longrightarrow$) We see from Lemma 3.9 that when $\lambda(\mu) = 1$, there exists a positive bounded solution.

($\Longleftarrow$) It follows from (6.2) that for $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$

$$\mathcal{E}^\mu(h\varphi) = \mathcal{E}^h(\varphi) + \mathcal{E}^\mu(h, h\varphi^2) = \mathcal{E}^h(\varphi) \leq C\mathcal{E}(\varphi), \quad C = \|h\|_\infty^2.$$

Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $0 \leq \varphi_n \uparrow 1$ m-a.e. and $\mathcal{E}(\varphi_n) \to 0$ as $n \to \infty$. Hence $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{E}^\mu(h) = 0$, which implies the criticality of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

(ii) $\lambda(\mu) > 1$ is equivalent to the boundedness of $\widetilde{g}^\mu(x)$, which is a positive solution. □

Lemma 6.2. *Suppose that $\Lambda(\mu) \geq 0$ and h is a non-trivial solution in $\mathcal{D}_e(\mathcal{E}^\mu)$. Then $|h|$ belongs to $\mathcal{H}_+^{\rm exc}(\mu)$.*

Proof. Since $0 \leq \mathcal{E}^\mu(|h|) \leq \mathcal{E}^\mu(h) = 0$, $\mathcal{E}^\mu(|h|) = 0$ and thus $|h|$ is also non-trivial solution. Hence this lemma follows from Lemma 4.3. □

Theorem 6.3. *Suppose that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite. If there exists a non-trivial solution h to $\mathcal{L}^\mu h = 0$ and a function ρ such that*

(i) $|h| \leq \rho$
(ii) *$(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable and its closure $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ is a recurrent Dirichlet form on $L^2(E; m)$,*

then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical.

Proof. Since $h\varphi \in \mathcal{D}(\mathcal{E})$ for $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$

$$\mathcal{E}^\mu(h\varphi) = \mathcal{E}^h(\varphi) + \mathcal{E}^\mu(h, h\varphi^2) = \mathcal{E}^h(\varphi) \leq \mathcal{E}^\rho(\varphi).$$

Since $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ is recurrent, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $0 \leq \varphi_n \uparrow 1$ m-a.e. and $\mathcal{E}^\rho(\varphi_n) \to 0$ as $n \to \infty$. Hence $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{E}^\mu(h) = 0$. Then $|h|$ is a non-trivial solution and belongs to $\mathcal{H}_+^{\rm exc}(\mu)$ by Lemma 6.2. Therefore, $1 \in \mathcal{D}_e(\mathcal{E}^{\mu,|h|})$ and $\mathcal{E}^{\mu,|h|}(1) = \mathcal{E}^\mu(|h|) = 0$, which implies the criticality of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$. □

By contraposition to Theorem 6.3, we have the next theorem, which tells us a Liouville-type property for Schrödinger forms.

Theorem 6.4. *Suppose that $\lambda(\mu) > 1$. If h is a solution satisfying the conditions (i), (ii) in Theorem 6.3, then $h \equiv 0$.*

We can give another proof of Theorem 6.4: Suppose that h is a solution satisfying (i), (ii) in Theorem 6.3. By the same argument in Theorem 6.3, $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{E}^\mu(h) = 0$. This assertion follows from Lemma 3.8.

Theorem 6.5. *Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent. If there exists a sign-changing bounded solution h, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is not positive semi-definite, that is, $\Lambda(\mu) < 0$.*

Proof. By the same argument as in the proof of Theorem 6.1, we see that $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{E}^\mu(h) = 0$. Suppose $\Lambda(\mu) \geq 0$ ($\Longleftrightarrow \lambda(\mu) \geq 1$). By Theorem 3.4, there exists a $g \in \mathcal{H}_+^{\mathrm{exc}}(\mu)$. Hence $h/g \in \mathcal{D}_e(\mathcal{E}^{\mu,g})$ and so $\mathcal{E}^{\mu,g}(h/g) = \mathcal{E}^\mu(h) = 0$. Hence, $h = Cg$, which is contradictory to that h is sign-changing. □

By the same argument as in Theorem 6.5, we have

Theorem 6.6. *If there exists a sign-changing locally bounded solution $h \in \mathcal{D}^\dagger_{loc}(\mathcal{E})$ to $\mathcal{L}^\mu h = 0$ and a function ρ satisfying*

(i) $|h| \leq \rho$
(ii) $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ *is closable on $L^2(E; m)$ and its closure $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ is a recurrent Dirichlet form,*

then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is not positive semi-definite, that is, $\Lambda(\mu) < 0$.

Example 6.7. Let $\mathcal{L}^\mu = (1/2)\Delta - \mu$ and $h \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ a solution to $\mathcal{L}^\mu h = 0$ satisfying the conditions (i), (ii) in Theorem 6.3 for ρ. Note that the transience and recurrence of Dirichlet forms are generally invariant by time change ([11, Theorem 6.2.3]), and so the recurrence of $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ on $L^2(E; m)$ is equivalent to that on $L^2(E; \rho^2 m)$. A sufficient condition for the Dirichlet form

$$\mathcal{E}^\rho(u) = \frac{1}{2}\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\rho^2 dx.$$

on $L^2(\mathbb{R}^d; \rho^2 dx)$ being recurrent is

$$\int_1^\infty \frac{r dr}{\int_{B(r)} \rho^2 dx} = \infty \tag{6.3}$$

([26, Theorem 3]). Then h is not sign-changing if $\lambda(\mu) \geq 1$ and in particular, identically 0 if $\lambda(\mu) > 1$. Theorem 1.7 in [4] and Theorem 2.4 in [12] treat cases when $|h(x)| \leq c|x|^{1-\frac{d}{2}}$ and $d = 3$, $|h(x_1, x_2, x_3)| \leq e^{-c|x_3|}$ respectively, which satisfy (6.3).

Theorem 6.8. *Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is recurrent. Let ρ be a positive function and $(\mathcal{E}^\rho, \mathcal{D}(\mathcal{E}^\rho))$ be another regular Dirichlet form with the same core $\mathcal{C}$ as $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Let h be a solution to $\mathcal{L}^\mu h = 0$. If $h\rho$ is bounded, then h is constant.*

Proof. Since

$$\mathcal{E}^\rho(h\varphi) = \mathcal{E}^{h\rho}(\varphi) + \mathcal{E}^\rho(h, h\varphi^2) = \mathcal{E}^{h\rho}(\varphi) \leq \|h\rho\|^2_\infty \mathcal{E}(\varphi),$$

h belongs to $\mathcal{D}_e(\mathcal{E}^\rho)$ and $\mathcal{E}^\rho(h) = 0$, and thus h is constant. □

Corollary 6.9. *Suppose that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is recurrent. Let $\mu\in\mathcal{K}_{loc}-\mathcal{K}_\infty(X^+)$ and $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite. If h is a positive bounded subsolution, then $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is critical.*

Example 6.10. ([30, Example 5.4]) Let X be the relativistic α-stable process with mass $m>0$, that is, the Lévy process on $\mathbb{R}^d$ with generator $\sqrt{-\Delta+m^2}-m$. The Dirichlet form generated by X is written as

$$\mathcal{E}(u)=\Big(\frac{m}{2\pi}\Big)^{(d+1)/2}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d}\frac{(u(x)-u(y))^2}{|x-y|^2}K_{(d+1)/2}(m|x-y|)dxdy,$$

Here $K_{(n+1)/2}$ is the modified Bessel function ([17, Theorem 7.12]). We see from its transition density (cf. [17, p. 183]) that if $d\le 2$, then $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is recurrent. Let $\mu\in\mathcal{K}_{\rm loc}-\mathcal{K}_\infty(X^+)$ such that $\lambda(\mu)\ge 1$. If there exists a bounded positive subsolution h to $\sqrt{-\Delta+m^2}-m-\mu$, then $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is critical and h is the ground state.

Example 6.11. Let X be the symmetric α-stable process on $\mathbb{R}^d$, the pure jump process generated by $(-\Delta)^{\alpha/2}$. The Dirichlet form generated by X is given by

$$\mathcal{E}(u)=K(\alpha,d)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy,$$

For a positive function ρ, define a form by

$$\mathcal{E}^\rho(u,v)=K(\alpha,d)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus d}\frac{(u(x)-u(y))(v(x)-v(y))\rho(x)\rho(y)}{|x-y|^{d+\alpha}}dxdy,$$

and assume $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E})\cap C_0(\mathbb{R}^d))$ is closable. If there exists a subsolution h satisfying that $|h\rho|\le c((1/|x|^\beta)\wedge 1)$, $\beta=(d-\alpha)/2$, then h is constant. More precisely, if $(\mathcal{E}^\rho,\mathcal{D}(\mathcal{E}^\rho))$ is transient, then $h\equiv 0$. Let $\mu\in\mathcal{K}_{\rm loc}-\mathcal{K}_\infty(X^+)$ with $\lambda(\mu)\ge 1$ and h a weak subsolution to $(-\Delta)^{\alpha/2}+\mu$. If $|h|\le c((1/|x|^\beta)\wedge 1)$, then h is not sign-changing, in particular, $h\equiv 0$ if $\lambda(\mu)=1$. The last statement is an extension of Example 6.7.

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