CRITICALITY OF SCHRÖDINGER FORMS AND RECURRENCE OF DIRICHLET FORMS

MASAYOSHI TAKEDA, TOSHIHIRO UEMURA

Abstract. Introducing the notion of extended Schrödinger spaces, we define the criticality and subcriticality of Schrödinger forms in the same manner as the recurrence and transience of Dirichlet forms, and give a sufficient condition for the subcriticality of Schrödinger forms in terms the bottom of spectrum. We define a subclass of Hardy potentials and prove that Schrödinger forms with potentials in this subclass are always critical, which leads us to optimal Hardy type inequality. We show that this definition of criticality and subcriticality is equivalent to that there exists an excessive function with respect to Schrödinger semigroup and its generating Dirichlet form through h-transform is recurrent and transient respectively. As an application, we can show the recurrence and transience of a family of Dirichlet forms by showing the criticality and subcriticaly of Schrödinger forms and show the other way around through h-transform, We give a such example with fractional Schrödinger operators with Hardy potential.

1. Introduction

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $X = (P_x, X_t, \zeta)$ be an m-symmetric Hunt process. We assume that X is irreducible, strong Feller and transient, in addition, that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ generated by X is regular on $L^2(E; m)$. For a Green-tight Kato measure μ ($\mu \in \mathcal{K}_\infty$ in notation), define a Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ by

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^\mu)(= \mathcal{D}(\mathcal{E})). \tag{1.1}$$

One of authors define in [28] the criticality or subcriticality for $\mathcal{E}^\mu$ through h-transform; let A_t^μ be the positive continuous additive functional with Revuz

2010 *Mathematics Subject Classification.* 60J46, 31C25, 31C05, 60J25.

The first author was supported in part by Grant-in-Aid for Scientific Research (No.18H01121(B)), Japan Society for the Promotion of Science. The second author was supported in part by Grant-in-Aid for Scientific Research (No.19K03552(C)), Japan Society for the Promotion of Science.

measure μ and $\{p_t^\mu\}_{t\geq 0}$ the Feynman-Kac semigroup defined by

$$p_t^\mu f(x) = E_x\left(e^{A_t^\mu} f(X_t)\right). \tag{1.2}$$

We introduce the space of p_t^μ-excessive functions by

$$\mathcal{H}^+(\mu) = \{h \mid h \text{ is quasi-continuous, } h > 0 \text{ q.e., } p_t^\mu h \leq h \text{ q.e.}\}. \tag{1.3}$$

Here the term "q.e." means "except on a set of zero capacity". Suppose that $\mathcal{H}^+(\mu)$ is not empty. Then for $h \in \mathcal{H}^+(\mu)$ the symmetric form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ on $L^2(E; h^2m)$ is defined by h-transform of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$:

$$\begin{cases} \mathcal{E}^{\mu,h}(u,u) = \mathcal{E}^\mu(hu, hu) \\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(E; h^2m) \mid hu \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{cases} \tag{1.4}$$

Li show in [17] that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ turns out to be a *quasi-regular Dirichlet form* on $L^2(E; h^2m)$. Consequently, if $\mathcal{H}^+(\mu)$ is not empty, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite, $\mathcal{E}^\mu(u,u) \geq 0$ for all $u \in \mathcal{D}(\mathcal{E}^\mu)$. The $L^2(E; h^2m)$-Markov semigroup $\{T_t^{\mu,h}\}_{t\geq 0}$ associated with $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is expressed by

$$T_t^{\mu,h} f(x) = \frac{1}{h(x)} p_t^\mu (hf)(x) \quad m\text{-a.e.}$$

In [17], [28], we define the subcriticality (resp. criticality) for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ as follows: $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be critical (resp. subcritical) if $\mathcal{H}^+(\mu)$ is not empty and $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is recurrent (resp. transient) for some $h \in \mathcal{H}^+(\mu)$. Besides, $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be supercritical if $\mathcal{H}^+(\mu)$ is empty. We show that these definitions are well-defined and obtain the following analytic criteria for the subcriticality and criticality ([28, Theorem 5.19]): Define $\lambda(\mu)$ by

$$\lambda(\mu) := \inf\left\{\mathcal{E}(u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}) \cap C_0(E),\ \int_E u^2 d\mu = 1\right\}. \tag{1.5}$$

(We simply write $a(u)$ for $a(u,u)$ for a symmetric form $a(u,v)$). Here, $\lambda(\mu)$ is regarded as the principal eigenvalue of the time changed process by A_t^μ. Then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, critical and supercritical, if and only if $\lambda(\mu) > 1$, $\lambda(\mu) = 1$ and $\lambda(\mu) < 1$ respectively. For establishing this analytic criteria, the assumption $\mu \in \mathcal{K}_\infty$ is crucial. Indeed, since for $\mu \in \mathcal{K}_\infty$ the extended Dirichlet space of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is compactly embedded in $L^2(E;\mu)$ ([30, Theorem 4.8]) and thus the minimizer of (1.5) exists. By the argument similar to one in the proof of [32, Theorem 3.1] we see that the minimizer belongs to $\mathcal{H}^+(\mu)$ if and only if $\lambda(\mu) \geq 1$, i.e.

$$\lambda(\mu) \geq 1 \iff \mathcal{H}^+(\mu) \neq \emptyset. \tag{1.6}$$

Let $\gamma(\mu)$ be the bottom of the Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$:

$$\gamma(\mu) = \inf\left\{\mathcal{E}^\mu(u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}) \cap C_0(E),\ \int_E u^2 dm = 1\right\}. \tag{1.7}$$

Since $\lambda(\mu) \geq 1$ is equivalent to $\gamma(\mu) \geq 0$ ([33, Lemma 2.2]), the following Allegretto-Piepenbrink-type theorem is obtained: For $\mu \in \mathcal{K}_\infty$

$$\gamma(\mu) \geq 0 \iff \mathcal{H}^+(\mu) \neq \emptyset. \tag{1.8}$$

This is a reason why for $\mu \in \mathcal{K}_\infty$ the criticality or subcriticality of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ are completely classified in terms of $\lambda(\mu)$.

In this paper, we treat more general class of measures including Hardy class of potentials, and define the criticality (resp. subcriticality) of associated Schrödinger forms in another way similar to the recurrence (resp. transience) of Dirichlet forms, without using the h-transform. More precisely, Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form. We assume that it is irreducible and transient throughout this paper. For a smooth Radon measure μ, define the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ as in (1.1). We assume that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite, i.e., $\gamma(\mu)$ in (1.7) is non-negative, and closable in $L^2(E; m)$. We denote its closure by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ (For a sufficient condition for the closability, see Lemma 3.1, Theorem 4.3 below).

The criticality and subcriticality of Schrödinger forms are extended notions of the recurrence and transience of Dirichlet forms. For characterizing the recurrence and transience of Dirichlet forms the extended Dirichlet spaces play a crucial role. In Schmuland [25] and [26], he extends the notion of extended Dirichlet spaces to positive semi-definite symmetric closed forms with *Fatou property* and show that positivity preserving forms have the Fatou property. We denote by $\mathcal{D}_e(\mathcal{E}^\mu)$ the extended space of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ and call it *extended Schrödinger space* (The definition of extended Schrödinger spaces is given in Section 2). On account of these facts, we easily imagine that the notion of extended Schrödinger spaces is available for characterizing the criticality and subcriticality of Schrödinger forms. Indeed, we define a form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is *subcritical* if there exists a bounded function $g \in L^1(E; m)$ strictly positive m-a.e. such that

$$\int_E |u| g dm \leq \sqrt{\mathcal{E}^\mu(u)}, \quad u \in \mathcal{D}_e(\mathcal{E}^\mu) \tag{1.9}$$

and is *critical* if there exists a function ϕ in $\mathcal{D}_e(\mathcal{E}^\mu)$ strictly positive m-a.e. such that $\mathcal{E}^\mu(\phi) = 0$. In other words, the function ϕ is the ground state of the Schrödinger operator $\mathcal{H}^\mu$ associated with $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, $(-\mathcal{H}^\mu u, v)_m = \mathcal{E}^\mu(u, v)$. We would like to emphasize that the space $\mathcal{D}_e(\mathcal{E}^\mu)$ contains the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$ of the Dirichlet space $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and ϕ does not belong to $\mathcal{D}_e(\mathcal{E})$ generally.

Let $\{T_t^\mu\}$ be the $L^2(E; m)$-semigroup associated with $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$. For a general smooth Radon measure μ, we do not know whether the semigroup T_t^μ is probabilistically expressed as

$$T_t^\mu f(x) = p_t^\mu f(x) \quad m\text{-a.e. }, \tag{1.10}$$

where the semigroup p_t^μ is defined in (1.2), and thus define the space of T_t^μ-excessive functions by

$$\mathcal{H}^+(\mu) = \{h \mid 0 < h < \infty \ m\text{-a.e.},\ T_t^\mu h \le h \ m\text{-a.e.}\}. \tag{1.11}$$

Here note that T_t^μ can be extended to an operator on the space of non-negative functions.

For $h \in \mathcal{H}^+(\mu)$ let $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ be the Dirichlet form on $L^2(E; h^2 m)$ defined by h-transform of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ as (1.4). We then show that if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, then the 0-resolvent $G^\mu g(x) = \int_0^\infty T_t^\mu g(x)dt$ of the function g in (1.9) belongs to $\mathcal{D}_e(\mathcal{E}^\mu) \cap \mathcal{H}^+(\mu)$ and $(\mathcal{E}^{\mu,G^\mu g}, \mathcal{D}(\mathcal{E}^{\mu,G^\mu g}))$ has the killing part, in particular, transient (Lemma 2.3, Lemma 2.11 below). Consequently, for any $h \in \mathcal{H}^+(\mu)$ $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient (Lemma 2.10).

Suppose that μ satisfies $\lambda(\mu) > 1$, where $\lambda(\mu)$ is the bottom of spectrum in (1.5). [1] We then show that $G^\mu\varphi$ is in $\mathcal{D}_e(\mathcal{E}^\mu) \cap \mathcal{H}^+(\mu)$ for any non-trivial, non-negative function $\varphi \in C_0(E)$, which implies $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical. Note that μ is not supposed to be Green-tight. Conversely, we can show if there exists a $h \in \mathcal{H}^+(\mu)$ such that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical.

We show that if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical, then the ground state ϕ in the definition of the criticality satisfies $T_t^\mu\phi = \phi$, in particular, $\phi \in \mathcal{H}^+(\mu)$, and $(\mathcal{E}^{\mu,\phi}, \mathcal{D}(\mathcal{E}^{\mu,\phi}))$ is recurrent. Moreover, every function $h \in \mathcal{H}^+(\mu)$ can be written as $h = c\phi,\ c > 0$. Conversely, if there exists a $h \in \mathcal{H}^+(\mu)$ such that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is recurrent, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical. In Section 6, we treat a family of Dirichlet forms defined in (1.17) $(0 \le \delta \le d - \alpha)$ below and show that it is recurrent if and only if $\delta = (d - \alpha)/2$.

Here we would like to make a comment on problems caused by treating a general smooth Radon measure instead of a Green-tight Kato measure: When $\lambda(\mu) = 1$, we do not know how to construct a function in $\mathcal{H}^+(\mu)$ and whether a Schrödinger form is subcritical or critical. In particular, we do not know whether the Allegretto-Piepenbrink-type theorem (1.8) holds or not (For recent results on Allegretto-Piepenbrink-type theorem, see [13], [15], [19]). Consequently, we cannot classify positive semi-definite Schrödinger forms to subcritical and critical ones in terms of $\lambda(\mu)$.

From Section 4, we treat the symmetric Hunt process X generated by the regular Dirichlet space $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. By the assumption on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, it is irreducible and transient. We assume, in addition, that it possesses the strong Feller property. As remarked above, we do not know whether T_t^μ is probabilistically expressed as (1.10). Suppose that μ is a smooth measure in local Kato class ($\mu \in \mathcal{K}_{\rm loc}$ in notation), that is, for any compact set $F \subset E$ the restriction of μ to F belongs to the Kato class $\mathcal{K}$ (Definition 4.1).[2] Using

[1]For a general smooth Radon measure μ we do not know whether $\lambda(\mu)$ is the principal eigenvalue or not.

[2]By the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, a measure in $\mathcal{K}_{\rm loc}$ is Radon.

facts in Albeverio and Ma [1] we can show that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E})\cap C_0(E))$ is closable and its semigroup T_t^μ has a probabilistic representation (1.10). As a result, we prove in Theorem 4.2 and Theorem 4.3 that the $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is regarded as the closed form generated by $p_t^\mu f(x)$. These facts can be extended to a smooth measure μ such that there exists a compact set K with $\mathrm{Cap}(K) = 0$ and $\mu \in \mathcal{K}_{\mathrm{loc}}^D$, $D = E\setminus K$, that is, for any compact set $F \subset D$ the restriction of μ to F belongs to the Kato class $\mathcal{K}$. Here Cap is the capacity associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Furthermore, we see that the closure of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E})\cap C_0(D))$ is identified with that of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E})\cap C_0(E))$.

We denote by $R(x, y)$ the 0-resolvent kernel of X and define the potential of a measure μ by

$$R\mu(x) = \int_E R(x, y)d\mu(y).$$

We introduce a subclass $\mathcal{K}_H$ of $\mathcal{K}_{\mathrm{loc}}$ as follows: A measure $\mu \in \mathcal{K}_{\mathrm{loc}}$ belongs to $\mu \in \mathcal{K}_H$ if μ satisfies that $R\mu$ is in $\mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap \mathscr{B}_{b,\mathrm{loc}}(E)$ and there exists an increasing sequence $\{K_n\}$ of compact sets such that $K_n \uparrow E$ and

$$\sup_n \iint_{K_n\times K_n^c} R(x, y)d\mu(x)d\mu(y) < \infty. \tag{1.12}$$

Here $\mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap \mathscr{B}_{b,\mathrm{loc}}(E)$ is the set of functions locally in $\mathcal{D}(\mathcal{E}) \cap \mathscr{B}_b(E)$. A measure $\mu \in \mathcal{K}_{\mathrm{loc}}$ of finite energy integral belongs to $\mathcal{K}_H$. For $\mu \in \mathcal{K}_H$ define a measure ν by $\nu = \mu/R\mu$. We then see that ν belongs to $\mathcal{K}_{\mathrm{loc}}$ and the Schrödinger form $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ can be defined. In Section 5, we show that for $\mu \in \mathcal{K}_H$, $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ is always critical. More precisely, $R\mu$ belongs to the extended Schrödinger space $\mathcal{D}_e(\mathcal{E}^\nu))$ and $\mathcal{E}^\nu(R\mu) = 0$, that is, $R\mu$ is a ground state of $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$. If μ is of finite energy integral, then $R\mu$ is in the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$; however, it does not hold generally. For this reason, it makes sense to introduce the extended Schrödinger space.

Denoting $\mathcal{L}$ the generator of X, we have

$$\mathcal{L}R\mu + R\mu \cdot \nu = -\mu + \mu = 0,$$

and a Hardy-type inequality

$$\int_E u^2 d\nu \le \mathcal{E}(u), \quad u \in \mathcal{D}(\mathcal{E}) \cap C_0(E) \tag{1.13}$$

(Fitzsimmons [10, Theorem 1.9]). Owing to the criticality of $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$, we see that the inequality (1.13) is optimal in the sense that the inequality (1.13) fails to hold if the constant of the left hand side is replaced by a constant bigger than 1. We learn from [22], [23] that the criticality of Schrödinger operators leads the optimality of Hardy-type inequalities. For example, if X is the absorbing Brownian motion on $(0, \infty)$ and $\mu(dx) = x^{-3/2}dx$, then μ belongs to $\mathcal{K}_H$ and ν equals $(1/4)x^{-2}dx$, which leads us to the classical Hardy inequality. The best constant $1/4$ appears naturally by calculating $\mu/R\mu$ and the constant $3/2$ is determined by the condition $x^{-p}dx \in \mathcal{K}_H$.

In Example 5.6, we treat the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ generated a transient symmetric α-stable process on $\mathbb{R}^d$ $(0 < \alpha < 2,\ \alpha < d)$ and consider the Schrödinger form with Hardy potential: for $u \in \mathcal{D}(\mathcal{E}^{(\alpha)} \cap C_0(\mathbb{R}^d)$

$$\mathcal{E}^{\mu^\delta}(u) = \mathcal{E}^{(\alpha)}(u) - \int_{\mathbb{R}^d} u^2 d\mu^\delta,$$

where $\mu^\delta(dx) = \kappa(\delta)/|x|^\alpha dx$ and

$$\kappa(\delta) = \frac{2^\alpha \Gamma\left(\frac{\delta+\alpha}{2}\right)\Gamma\left(\frac{d-\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{d-\delta-\alpha}{2}\right)} \quad (0 \le \delta \le d-\alpha). \tag{1.14}$$

Noting that $\mathrm{Cap}(\{0\}) = 0$ and μ^δ is in the local Kato class on $\mathbb{R}^d \setminus \{0\}$, we see from Theorem 4.3 that $(\mathcal{E}^{\mu^\delta}, C_0^\infty(\mathbb{R}^d))$ is closable, Denote by $\mathcal{D}(\mathcal{E}^{\mu^\delta})$ and $\mathcal{D}_e(\mathcal{E}^{\mu^\delta})$ the closure and its extended Schrödinger space. Let $\delta^* = (d-\alpha)/2$ and $\kappa^* = \kappa(\delta^*)$. Then it is known that

$$\kappa^* = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{4}\right)^2}{\Gamma\left(\frac{d-\alpha}{4}\right)^2} \tag{1.15}$$

is the best constant of the Hardy inequality: for $u \in \mathcal{D}(\mathcal{E}^{(\alpha)})$

$$\kappa^* \int_{\mathbb{R}^d} u^2 \frac{1}{|x|^\alpha} dx \le \mathcal{E}^{(\alpha)}(u) \tag{1.16}$$

(e.g. [4]). We show in this example that the power δ^* is determined by the condition $|x|^{-p}dx \in \mathcal{K}_H$ and the best constant κ^* comes from the calculation of $\mu^{\delta^*}/R\mu^{\delta^*}$, where R is the resolvent kernel of the symmetric α-stable process. We have to say that the argument in this section is strongly motivated by that in Miura [20], [21].

If X has no killing inside, the semigroup $\widetilde{p}_t^\nu$ defined by

$$\widetilde{p}_t^\nu f(x) = E_x\left(\frac{1}{R\mu(X_0)} \exp\left(\int_0^t \frac{dA_s^\mu}{R\mu(X_s)}\right) R\mu(X_t) f(X_t)\right)$$

turns out to be Markovian, $\widetilde{p}_t^\nu 1(x) \le 1$ and the Dirichlet form generated by the semigroup $\{\widetilde{p}_t^\nu\}$ equals $(\mathcal{E}^{\nu,R\mu}, \mathcal{D}(\mathcal{E}^{\nu,R\mu}))$. The criticality of $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ implies the recurrence of $(\mathcal{E}^{\nu,R\mu}, \mathcal{D}(\mathcal{E}^{\nu,R\mu}))$.

In Section 6, we consider again the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ and the Schrödinger form $(\mathcal{E}^{\mu^\delta}, \mathcal{D}(\mathcal{E}^{\mu^\delta}))$ in Example 5.6. We know from [4, Figure 1] that

$$\kappa(\delta) < \kappa^* \text{ for } \delta \ne \delta^* \text{ and } \kappa(\hat{\delta}) = \kappa(\delta) \text{ for } \hat{\delta} = d - \alpha - \delta.$$

Moreover, we see from these facts that

$$\lambda(\mu^{\delta^*}) = 1 \text{ and } \lambda(\mu^\delta) > 1 \text{ for } \delta \ne \delta^*.$$

Moreover, it is known in [4, Theorem 3.1, Theorem 5.4]) that the function $|x|^{-\delta}$, $0 \le \delta \le \delta^*$, satisfies $p_t^{\mu^\delta}|x|^{-\delta} = |x|^{-\delta}$ and the Dirichlet form defined by h-transform of $\mathcal{E}^{\mu^\delta}$ by $|x|^{-\delta}$ is written as

$$\mathcal{E}^{\mu^\delta,|x|^{-\delta}}(u) = \frac{1}{2}\mathcal{A}(d,\alpha)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}|x|^\delta|y|^\delta}dxdy. \tag{1.17}$$

In particular, we see that for $0 \le \delta \le \delta^*$, $|x|^{-\delta}$ belongs to $\mathcal{H}^+(\mu^\delta)$. In this section, we give another proof that $(\mathcal{E}^{\mu^{\delta^*}}, \mathcal{D}(\mathcal{E}^{\mu^{\delta^*}}))$ is critical in our sense. Indeed, we shall prove that if $\delta = \delta^*$ the corresponding Dirichlet form in (1.17) is recurrent by showing that 1 is contains in the extended Dirichlet space $\mathcal{D}_e(\mathcal{E}^{\mu^{\delta^*},|x|^{-\delta^*}})$ and $\mathcal{E}^{\mu^{\delta^*},|x|^{-\delta^*}}(1) = 0$. As a result, we see that the function $|x|^{-\delta^*}$ belongs to the extended Schrödinger space of $\mathcal{D}_e(\mathcal{E}^{\mu^{\delta^*}})$ and $\mathcal{E}^{\mu^{\delta^*}}(|x|^{-\delta^*}) = 0$, that is, $|x|^{-\delta^*}$ is a ground state. Note that $|x|^{-\delta^*}$ does not belong to the extended Dirichlet space $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$. Indeed, if $|x|^{-\delta^*}$ belongs to $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$, it must be in $L^2(|x|^{-\alpha}dx)$ by the Hardy inequality (1.16). In other words, if $\delta = \delta^*$, then the extended Schrödinger space strictly contains the extended Dirichlet space. On the other hand, if $\delta \ne \delta^*$, the extended Schrödinger space is identical with $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ (Lemma 3.1), which does not contain $|x|^{-\delta}$ by the same reason above.

We show that for $0 \le \delta < \delta^*$ $(\mathcal{E}^{\mu^\delta}, \mathcal{D}(\mathcal{E}^{\mu^\delta}))$ is subcritical. As a result, we see that $(\mathcal{E}^{\mu^\delta,|x|^{-\delta}}, \mathcal{D}(\mathcal{E}^{\mu^\delta,|x|^{-\delta}}))$ is transient. Finally we see that by the *inversion property* between X^δ and $X^{\hat\delta}$ $(\mathcal{E}^{\mu^\delta}, \mathcal{D}(\mathcal{E}^{\mu^\delta}))$ is transient for $\delta^* < \delta \le d-\alpha$, where X^δ and $X^{\hat\delta}$ are Markov processes generated by the Dirichlet form in (1.17) for δ and $\hat\delta$ respectively (Remark 6.4).

2. Schrödinger forms

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(X;m)$. We denote by $u \in \mathcal{D}_{loc}(\mathcal{E})$ if for any relatively compact open set D there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. Let $\mathcal{D}_e(\mathcal{E})$ be the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists a sequence $\{u_n\}$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n,m\to\infty}\mathcal{E}(u_n - u_m) = 0$ and $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the sequence $\{u_n\}$ an *approximating sequence* of $u \in \mathcal{D}_e(\mathcal{E})$. We can extend the form $\mathcal{E}$ to $\mathcal{D}_e(\mathcal{E})$: for approximating sequences $\{u_n\}$ and $\{v_n\}$ of u and $v \in \mathcal{D}_e(\mathcal{E})$

$$\mathcal{E}(u,v) = \lim_{n\to\infty}\mathcal{E}(u_n, v_n).$$

Definition 2.1. (1) A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ is said to be *transient* if there exists a bounded function $g \in L^1(E;m)$ strictly positive

m-a.e. such that

$$\int_E |u| g dm \leq \sqrt{\mathcal{E}(u)}, \quad u \in \mathcal{D}(\mathcal{E}).$$

(2) A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ is said to be *recurrent* if the constant function 1 belongs to $\mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(1) = 0$.

For other characterizations of transience and recurrence, see [12, Theorem 1.6.2, Theorem 1.6.3]. Throughout this paper, we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient and irreducible, i.e., the $L^2(E;m)$-Markov semigroup $\{T_t\}$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies that if an m-measurable set $A \subset E$ satisfies $T_t(1_A f) = 1_A T_t f$ m-a.e. for any $f \in L^2(E;m)$ and any $t > 0$, then $m(A) = 0$ or $m(A^c) = 0$.

We define the *(1-)capacity* Cap associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ as follows: for an open set $O \subset E$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u) \mid u \in \mathcal{D}(\mathcal{E}), u \geq 1, \ m\text{-a.e. on } O\}$$

and for a Borel set $A \subset E$,

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) \mid O \text{ is open}, O \supset A\},$$

where $\mathcal{E}_1(u) = \mathcal{E}(u) + (u,u)_m$.

A statement depending on $x \in E$ is said to hold q.e. on E if there exists a set $N \subset E$ of zero capacity such that the statement is true for every $x \in E \setminus N$. "q.e." is an abbreviation of "quasi-everywhere". A real valued function u defined q.e. on E is said to be *quasi-continuous* if for any $\epsilon > 0$ there exists an open set $G \subset E$ such that $\mathrm{Cap}(G) < \epsilon$ and $u|_{E \setminus G}$ is finite and continuous. Here, $u|_{E \setminus G}$ denotes the restriction of u to $E \setminus G$. Each function u in $\mathcal{D}_e(\mathcal{E})$ admits a quasi-continuous version $\tilde{u}$, that is, $u = \tilde{u}$ m-a.e. In the sequel, we always assume that every function $u \in \mathcal{D}_e(\mathcal{E})$ is represented by its quasi-continuous version.

We call a positive Borel measure μ on E *smooth* if it satisfies

(i) μ charges no set of zero capacity,
(ii) there exists an increasing sequence $\{F_n\}$ of closed sets such that
   a) $\mu(F_n) < \infty, \ n = 1, 2, \ldots,$
   b) $\lim_{n \to \infty} \mathrm{Cap}(K \setminus F_n) = 0$ for any compact set K.

Let μ be a positive smooth Radon measure. We define a Schrödinger form on $L^2(E;m)$ by

$$\mathcal{E}^\mu(u,v) = \mathcal{E}(u,v) - \int_E uv d\mu, \quad u, v \in \mathcal{D}(\mathcal{E}) \cap C_0(E). \tag{2.1}$$

We assume that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite, $\mathcal{E}^\mu(u) \geq 0$ for $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$, and closable. We denote by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ its closure. By the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E}^\mu)$ and

$$\int_E u^2 d\mu \le \mathcal{E}(u), \quad \mathcal{E}^\mu(u) = \mathcal{E}(u) - \int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}).$$

The Schrödinger form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is expressed by the non-positive self-adjoint operator $\mathcal{H}^\mu$ as follows:

$$\mathcal{E}^\mu(u) = \|\sqrt{-\mathcal{H}^\mu} u\|_2^2, \quad \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\sqrt{-\mathcal{H}^\mu}).$$

Then the $L^2(E;m)$-strong continuous semigroup $T_t^\mu := \exp(t\mathcal{H}^\mu)$ is defined and it is contractive, $\|T_t^\mu\|_2 \le 1$, where $\|\cdot\|_2$ is the operator norm on $L^2(E;m)$.

A densely defined, closed, positive semi-definite symmetric bilinear form $(a, \mathcal{D}(a))$ is said to be *positive preserving* if for $u \in \mathcal{D}(a)$ $|u|$ belongs to $\mathcal{D}(a)$ and $a(|u|) \le a(u)$. It follow from [8, Lemma 1.3.4] that the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive preserving because $\mathcal{E}^\mu(|u|) \le \mathcal{E}^\mu(u)$ for $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$. As a result, we see from [26, Proposition 2] that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ has the *Fatou property*, i.e., if $\{u_n\} \subset \mathcal{D}(\mathcal{E}^\mu)$ satisfies $\sup_n \mathcal{E}^\mu(u_n) < \infty$ and $u_n \to u \in \mathcal{D}(\mathcal{E}^\mu)$ m-a.e., then $\liminf_{n\to\infty} \mathcal{E}^\mu(u_n) \ge \mathcal{E}^\mu(u)$. Hence, following [25], we can define a space $\mathcal{D}_e(\mathcal{E}^\mu)$ in the way similar to the extended Dirichlet space: An m-measurable function u with $|u| < \infty$ m-a.e. is said to be in $\mathcal{D}_e(\mathcal{E}^\mu)$ if there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E}^\mu)$ the *extended Schrödinger space* of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ and the sequence $\{u_n\}$ an *approximating sequence* of u. For $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ and an approximating sequence $\{u_n\}$ of u define

$$\mathcal{E}^\mu(u) = \lim_{n\to\infty} \mathcal{E}^\mu(u_n). \tag{2.2}$$

We can give another definition of the extended Schrödinger space through h-transform. We introduce the space of T_t^μ-excessive functions:

$$\mathcal{H}^+(\mu) = \{h \mid 0 < h < \infty \ m\text{-a.e.}, \ T_t^\mu h \le h \ m\text{-a.e.}\}. \tag{2.3}$$

For $h \in \mathcal{H}^+(\mu)$ define the Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ by

$$\begin{cases} \mathcal{E}^{\mu,h}(u,v) = \mathcal{E}^\mu(uh, uh), \\ \mathcal{D}(\mathcal{E}^{\mu,h}) = \left\{u \in L^2(E; h^2 m) \mid uh \in \mathcal{D}(\mathcal{E}^\mu)\right\}, \end{cases} \tag{2.4}$$

and let $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ be the extended Dirichlet space of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. We then see that $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ is equivalent to $u/h \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$ ([28, Lemma 2.8]).

We define the criticality and subcriticality of Schrödinger forms in the way similar to the recurrence and transience of Dirichlet forms.

Definition 2.2. Let μ be a smooth Radon measure and $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ the positive semi-definite Schrödinger form.

(1) $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *subcritical* if there exists a bounded function g in $L^1(E;m)$ strictly positive m-a.e. such that

$$\int_E |u| g dm \le \sqrt{\mathcal{E}^\mu(u)}, \quad u \in \mathcal{D}_e(\mathcal{E}^\mu). \tag{2.5}$$

(2) $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *critical* if there exists a function ϕ in $\mathcal{D}_e(\mathcal{E}^\mu)$ strictly positive m-a.e. such that $\mathcal{E}^\mu(\phi) = 0$. The function ϕ is said to be the *ground state*.

Define the operator G^μ on a positive function f by

$$G^\mu f(x) = \int_0^\infty T_t^\mu f(x) dt \ (\leq +\infty).$$

Lemma 2.3. *Let g be the function in Definition 2.2 (1). Then $G^\mu g$ belongs to $\mathcal{D}_e(\mathcal{E}^\mu)$.*

Proof. By the same argument as in [12, Lemma 1.5.3] we have the next inequality: for any non-negative $f \in L^1(E;m) \cap L^2(E;m)$

$$\sup_{u \in \mathcal{D}(\mathcal{E}^\mu)} \frac{(|u|, f)}{\sqrt{\mathcal{E}^\mu(u)}} = \sqrt{\int_E f G^\mu f \, dm} \ (\leq +\infty).$$

Hence, the equation (2.5) implies $\int_E g G^\mu g \, dm \leq 1$. Moreover, by the same argument as in Theorem 1.5.4 (i) in [12] we see that $G^\mu g$ belongs to $\mathcal{D}_e(\mathcal{E}^\mu)$. □

Lemma 2.4. *Suppose $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical and let ϕ be the ground state. Then ϕ belongs to $\mathcal{H}^+(\mu)$, more precisely, $T_t^\mu \phi = \phi$ m-a.e.*

Proof. We see from Lemma 2.8 below. □

Remark 2.5. If $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ satisfies $\mathcal{E}^\mu(h) = 0$, then h is a weak solution to $\mathcal{H}^\mu h = 0$ in the sense that $\mathcal{E}^\mu(h, \varphi) = 0$ for any $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ because $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite. Moreover, if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical and ϕ is a ground state, then the function h is written as $h = c\phi$ (c is a constant), in particular, not sign-changing. Indeed, by Theorem 2.12, the Dirichlet form $(\mathcal{E}^{\mu,\phi}, \mathcal{D}(\mathcal{E}^{\mu,\phi}))$ is recurrent and

$$\mathcal{E}^{\mu,\phi}(h/\phi) = \mathcal{E}^\mu(h) = 0,$$

which implies h/ϕ is a constant m-a.e. by [14, Theorem 1].

Lemma 2.6. *If $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\{u_n\}_{n=1}^\infty$ is an approximating sequence of u, then $\lim_{n\to\infty} \mathcal{E}^\mu(u - u_n) = 0$.*

Proof. For any $\varepsilon > 0$ there exists N such that $\mathcal{E}^\mu(u_m - u_n) \leq \varepsilon$ for $m, n \geq N$. For a fixed n, $\{u_m - u_n\}_{m=1}^\infty$ is an approximating sequence of $u - u_n$ and so

$$\mathcal{E}^\mu(u - u_n) = \lim_{m\to\infty} \mathcal{E}^\mu(u_m - u_n) \leq \varepsilon, \quad n \geq N.$$

□

Lemma 2.7. *If $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, then $(\mathcal{D}_e(\mathcal{E}^\mu), \mathcal{E}^\mu)$ is a Hilbert space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence with respect to $\mathcal{E}^\mu$. Then $\{u_n\}$ is also a Cauchy sequence in $L^1(E;gm)$ by (2.5) and so there exists a subsequence $\{u_k\}$ of $\{u_n\}$ converges to $u \in L^1(E;gm)$ m-a.e. Hence u belongs to $\mathcal{D}_e(\mathcal{E}^\mu)$ and $\{u_k\}$ converges to u in $\mathcal{E}^\mu$ by Lemma 2.6. Since $\{u_n\}$ be a Cauchy sequence in $\mathcal{E}^\mu$, $\{u_n\}$ itself converges to u in $\mathcal{E}^\mu$. □

We can show in the same argument as in [31, Lemma 3.13] that the semi-group T_t^μ, $t > 0$ can be extended to an operator form $\mathcal{D}_e(\mathcal{E}^\mu)$ to itself. For $u \in \mathcal{D}(\mathcal{E}^\mu)$

$$\frac{1}{t}\|u - T_t^\mu u\|_2^2 = \frac{1}{t}(u - T_t^\mu u, u) - \frac{1}{t}(u - T_t^\mu u, T_t^\mu u) \leq \frac{1}{t}(u - T_t^\mu u, u)$$

because

$$(u - T_t^\mu u, T_t^\mu u) = \int_0^\infty \left(1 - e^{-\lambda t}\right) e^{-\lambda t} d(E_\lambda u, u) \geq 0$$

by the spectral decomposition theorem, $-H^\mu = \int_0^\infty \lambda dE_\lambda$. Here $\{E_\lambda\}_{\lambda \geq 0}$ is a resolution of the identity. Hence we have

$$\frac{1}{t}\|u - T_t^\mu u\|_2^2 \leq \mathcal{E}^\mu(u).$$

Since

$$\begin{aligned} \frac{1}{t}(u - T_t^\mu u, u) &= \int_0^\infty \left(\frac{1 - e^{-\lambda t}}{t}\right) d(E_\lambda u, u) \\ &\leq \int_0^\infty \lambda d(E_\lambda u, u) \\ &= \mathcal{E}^\mu(u). \end{aligned}$$

For $u \in \mathcal{D}_e(\mathcal{E}^\mu)$, let $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E}^\mu)$ be an approximating sequence of u. Then

$$\begin{aligned} \frac{1}{t}\|(u_n - T_t^\mu u_n) - (u_m - T_t^\mu u_m)\|_2^2 &= \frac{1}{t}\|(u_n - u_m) - T_t^\mu(u_n - u_m)\|_2^2 \\ &\leq \mathcal{E}^\mu(u_n - u_m), \end{aligned}$$

and thus $(u_n - T_t^\mu u_n)$ converges to a $v \in L^2(E;m)$ strongly. Let $\{u'_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E}^\mu)$ be another approximating sequence of u. Then

$$\begin{aligned} &\frac{1}{t}\|(u_n - T_t^\mu u_n) - (u'_n - T_t^\mu u'_n)\|_2^2 \leq \mathcal{E}^\mu(u_n - u'_n) \\ &\quad = \mathcal{E}^\mu(u_n) + \mathcal{E}^\mu(u'_n) - 2\mathcal{E}^\mu(u_n, u'_n) \\ &\quad \longrightarrow \mathcal{E}^\mu(u) + \mathcal{E}^\mu(u) - 2\mathcal{E}^\mu(u) = 0 \end{aligned}$$

as $n \to \infty$. Hence the function $v \in L^2(E;m)$ is independent of the choice of the approximating sequence.

Using the spectral decomposition theorem again, we have for $u \in \mathcal{D}(\mathcal{E}^\mu)$

$$\mathcal{E}^\mu(T_t^\mu u) = \int_0^\infty \lambda \left(e^{-2\lambda t}\right)^2 d(E_\lambda u, u) \le \int_0^\infty \lambda d(E_\lambda u, u) \le \mathcal{E}^\mu(u),$$

and $\mathcal{E}^\mu(T_t^\mu u_n - T_t^\mu u_m) \le \mathcal{E}^\mu(u_n - u_m)$. There exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ such that $T_t^\mu u_{n_k} = u_{n_k} - (u_{n_k} - T_t^\mu u_{n_k})$ converges to $u - v$ m-a.e. Hence, $u - v \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $\{T_t^\mu u_{n_k}\}_{k=1}^\infty$ is an approximating sequence of $u - v$. The semigroup T_t^μ can be extended to $\mathcal{D}_e(\mathcal{E}^\mu)$ by $T_t^\mu u = u - v$. We then see that $\lim_{n\to\infty} \|(u_n - T_t^\mu u_n) - (u - T_t^\mu u)\|_2 = 0$ and

$$\begin{aligned}\frac{1}{t}\|u - T_t^\mu u\|_2^2 = \frac{1}{t}\|v\|_2^2 &= \lim_{n\to\infty} \frac{1}{t}\|u_n - T_t^\mu u_n\|_2^2 \\ &\le \lim_{n\to\infty} \mathcal{E}^\mu(u_n) = \mathcal{E}^\mu(u).\end{aligned}$$

Hence, we have

Lemma 2.8. *The semigroup T_t^μ can be uniquely extended to a linear operator on $\mathcal{D}_e(\mathcal{E}^\mu)$ and for $u \in \mathcal{D}_e(\mathcal{E}^\mu)$*

$$\frac{1}{t}\|u - T_t^\mu u\|_2^2 \le \mathcal{E}^\mu(u).$$

The argument similar to that in the proof of [12, Lemma 1.5.4] leads us to the next lemma.

Lemma 2.9. *For $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ and $w \in \mathcal{D}(\mathcal{E}^\mu)$*

$$\lim_{t\downarrow 0} \frac{1}{t}(u - T_t^\mu u, w) = \mathcal{E}^\mu(u, w).$$

Proof. For $u, v \in \mathcal{D}(\mathcal{E}^\mu)$

$$\begin{aligned}\left|\frac{1}{t}(u - T_t^\mu u, w)\right| &\le \left(\frac{1}{t}(u - T_t^\mu u, u)\right)^{1/2} \left(\frac{1}{t}(w - T_t^\mu w, w)\right)^{1/2} \\ &\le \mathcal{E}^\mu(u)^{1/2}\mathcal{E}^\mu(w)^{1/2}.\end{aligned} \tag{2.6}$$

Let $\{u_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{E}^\mu)$ be an approximating sequence of u. Noting that the inequality (2.6) can be extended to $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ by the argument before Lemma 2.8, we have

$$\begin{aligned}\left|\frac{1}{t}(u - T_t^\mu u, w) - \frac{1}{t}(u_n - T_t^\mu u_n, w)\right| &= \left|\frac{1}{t}(u - u_n - T_t^\mu(u - u_n), w)\right| \\ &\le \mathcal{E}^\mu(u - u_n)^{1/2}\mathcal{E}^\mu(w)^{1/2},\end{aligned}$$

equivalently

$$\begin{aligned}\frac{1}{t}(u_n - T_t^\mu u_n, w) - \mathcal{E}^\mu(u - u_n)^{1/2}\mathcal{E}^\mu(w)^{1/2} &\le \frac{1}{t}(u - T_t^\mu u, w) \\ &\le \frac{1}{t}(u_n - T_t^\mu u_n, w) + \mathcal{E}^\mu(u - u_n)^{1/2}\mathcal{E}^\mu(w)^{1/2}.\end{aligned}$$

Hence we have

$$\begin{aligned}\mathcal{E}^\mu(u_n,w)-\mathcal{E}^\mu(u-u_n)^{1/2}\mathcal{E}^\mu(w)^{1/2} &\le \varliminf_{t\downarrow 0}\frac{1}{t}(u-T_t^\mu u,w)\\ &\le \varlimsup_{t\downarrow 0}\frac{1}{t}(u-T_t^\mu u,w) \qquad (2.7)\\ &\le \mathcal{E}^\mu(u_n,w)+\mathcal{E}^\mu(u-u_n)^{1/2}\mathcal{E}^\mu(w)^{1/2}.\end{aligned}$$

The both sides of (2.7) tend to $\mathcal{E}^\mu(u,w)$ as $n\to\infty$ and the proof is completed. □

Lemma 2.10. *Suppose $\mathcal{H}^+(\mu)$ is not empty. If $(\mathcal{E}^{\mu,h},\mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient for some $h\in\mathcal{H}^+(\mu)$, then so is for any $h\in\mathcal{H}^+(\mu)$.*

Proof. First note that by the irreducibility of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, $(\mathcal{E}^{\mu,h},\mathcal{D}(\mathcal{E}^{\mu,h}))$ is either transient or recurrent. Suppose that for $h_1,h_2\in\mathcal{H}^+(\mu)$, $(\mathcal{E}^{\mu,h_1},\mathcal{D}(\mathcal{E}^{\mu,h_1}))$ is transient and $(\mathcal{E}^{\mu,h_2},\mathcal{D}(\mathcal{E}^{\mu,h_2}))$ is recurrent. Then it follows that there exists $g>0$ such that

$$\int_E |u|gh_1^2dm \le \sqrt{\mathcal{E}^{\mu,h_1}(u)},\quad u\in\mathcal{D}_e(\mathcal{E}^{\mu,h_1}). \qquad (2.8)$$

Let $\{\psi_n\}\subset\mathcal{D}(\mathcal{E}^{\mu,h_2})$ be an approximating sequence 1 and $\mathcal{E}^{\mu,h_2}(\psi_n)\to 0$ as $n\to\infty$. Noting that $(h_2/h_1)\psi_n=(h_2\psi_n)/h_1\in\mathcal{D}_e(\mathcal{E}^{\mu,h_1})$, we have

$$\begin{aligned}\int_E (h_2/h_1)|\psi_n|gh_1^2dm &\le \sqrt{\mathcal{E}^{\mu,h_1}((h_2/h_1)\psi_n))}=\sqrt{\mathcal{E}^\mu(h_2\psi_n)} \qquad (2.9)\\ &=\sqrt{\mathcal{E}^{\mu,h_2}(\psi_n)}\to 0,\end{aligned}$$

which is contradictory because

$$\liminf_{n\to\infty}\int_E (h_2/h_1)|\psi_n|gh_1^2dm \ge \int_E h_1h_2gdm>0.$$

□

Lemma 2.11. *If $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ is subcritical, then the function $G^\mu g$ in Lemma 2.3 belongs to $\mathcal{H}^+(\mu)$ and $(\mathcal{E}^{\mu,G^\mu g},\mathcal{D}(\mathcal{E}^{\mu,G^\mu g}))$ is transient.*

Proof. The function $G^\mu g\in\mathcal{D}_e(\mathcal{E}^\mu)$ belongs to $\mathcal{H}^+(\mu)$. Indeed, let

$$S_T^\mu g(x)=\int_0^T T_s^\mu gds.$$

Then

$$T_t^\mu S_T^\mu g(x)=\int_0^{T+t} T_s^\mu gds-\int_0^t T_s^\mu gds$$

and by $T\to\infty$

$$T_t^\mu G^\mu g(x)=G^\mu g-\int_0^t T_s^\mu gds\le G^\mu g. \qquad (2.10)$$

Put $v = G^\mu g$. Then since

$$\mathcal{E}^{\mu,v}(1) = \mathcal{E}^\mu(v) = \int_E gG^\mu g dm > 0,$$

$(\mathcal{E}^{\mu,v}, \mathcal{D}(\mathcal{E}^{\mu,v}))$ is transient, more precisely, has the killing part. □

Theorem 2.12. i) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical if and only if there exists $h \in \mathcal{H}^+(\mu)$ such that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient.*
ii) *$(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical if and only if there exists $h \in \mathcal{H}^+(\mu)$ such that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is recurrent.*

Proof. i) The proof of "only if" part is given in Lemma 2.11. Let h be a function in $\mathcal{H}^+(\mu)$ such that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient. Then there exists a bounded function $g' \in L^1(E; h^2 m)$ strictly positive m-a.e. such that for $u \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$

$$\int_E |u| g' h^2 dm \le \sqrt{\mathcal{E}^{\mu,h}(u)}.$$

Putting $v = hu$, we have for $v \in \mathcal{D}_e(\mathcal{E}^\mu)$

$$\int_E |v| g' h dm \le \sqrt{\mathcal{E}^\mu(v)}.$$

Let $\eta \in L^1(E; m)$ with $0 < \eta \le 1$, m-a.e. and define $g = ((g'h) \wedge 1)\eta$. Then g is a bounded function in $L^1(E; m)$ positive m-a.e. such that

$$\int_E |v| g dm \le \sqrt{\mathcal{E}^\mu(v)}, \quad \forall v \in \mathcal{D}_e(\mathcal{E}^\mu).$$

ii) Let ϕ be a function in Definition 2.2 (2). Then we see from Lemma 2.4 that $\phi \in \mathcal{D}_e(\mathcal{E}^\mu) \cap \mathcal{H}^+(\mu)$. Hence $1 \in \mathcal{D}_e(\mathcal{E}^{\mu,\phi})$ and

$$\mathcal{E}^{\mu,\phi}(1) = \mathcal{E}^\mu(\phi) = 0.$$

Suppose that for an $h \in \mathcal{H}^+(\mu)$ $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is recurrent. Then $1 \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$ and $\mathcal{E}^{\mu,h}(1) = 0$. Hence h is in $\mathcal{D}_e(\mathcal{E}^\mu)$ and satisfies $\mathcal{E}^\mu(h) = 0$, that is, a ground state by Remark 2.5. □

3. Analytic Criterion for Subcriticality

We define a function space

$$\mathcal{L} = \left\{ f \;\Big|\; \left| \int_E |f| \varphi dm \right| \le C \mathcal{E}(\varphi)^{1/2} \text{ for any } \varphi \in \mathcal{D}_e(\mathcal{E}) \right\}. \tag{3.1}$$

In this section, we make an assumption:

$$1_K \in \mathcal{L} \quad \text{for any compact set } K. \tag{3.2}$$

Note that $\mathcal{D}(\mathcal{E}) \cap C_0(E) \subset \mathcal{L}$ by the assumption If the Hunt process X generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the strong Feller property, then the assumption above is fulfilled by the Green-boundedness of $1_K m$, $\|G1_K\|_\infty < \infty$ ([**?**, Proposition 2.2]) and the inequality

$$\int_E |\psi|\varphi dm \leq \|G|\psi|\|_\infty^{1/2} \cdot \left(\int_E |\psi| dm\right)^{1/2} \mathcal{E}(\varphi)^{1/2}. \tag{3.3}$$

We define

$$\lambda(\mu) = \inf\left\{\mathcal{E}(u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}) \cap C_0(E),\ \int_E u^2 d\mu = 1\right\}. \tag{3.4}$$

Lemma 3.1. *If $\lambda(\mu) > 1$, then there exists a positive constant c such that*

$$c \cdot \mathcal{E}(u) \leq \mathcal{E}^\mu(u) \leq \mathcal{E}(u),\ \ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Proof. By the definition of $\lambda(\mu)$

$$\int_E u^2 d\mu \leq \frac{1}{\lambda(\mu)} \cdot \mathcal{E}(u),\ \ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$$

and thus

$$\mathcal{E}^\mu(u) = \mathcal{E}(u) - \int_E u^2 d\mu \geq \mathcal{E}(u) - \frac{1}{\lambda(\mu)}\mathcal{E}(u) = \left(1 - \frac{1}{\lambda(\mu)}\right)\mathcal{E}(u).$$

□

We see from Lemma 3.1 that if $\lambda(\mu) > 1$, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite and closable. Furthermore, its extended Schrödinger space $\mathcal{D}_e(\mathcal{E}^\mu)$ equals the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$ and $(\mathcal{D}_e(\mathcal{E}^\mu), \mathcal{E}^\mu)$ is a Hilbert space.

Lemma 3.2. *Suppose $\lambda(\mu) > 1$. For $f \in \mathcal{L}$, there exists an $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ such that for any $\varphi \in \mathcal{D}_e(\mathcal{E}^\mu)$*

$$\mathcal{E}^\mu(h, \varphi) = \int_E f\varphi dm.$$

Proof. Since $(\mathcal{D}_e(\mathcal{E}^\mu), \mathcal{E}^\mu)$ is a Hilbert space, the Riesz theorem leads us to this lemma. □

Lemma 3.3. *Suppose $\lambda(\mu) > 1$. If $f \in \mathcal{L}$ is non-negative, then $h \in \mathcal{D}_e(\mathcal{E}^\mu)$ in Lemma 3.2 is also non-negative.*

Proof. Since $\mathcal{E}^\mu(h, \psi) = \int_E f\psi dm \geq 0$ for any non-negative $\psi \in \mathcal{D}_e(\mathcal{E}^\mu)$, $\mathcal{E}^\mu(h, h^-) \geq 0$ and so $\mathcal{E}^\mu(h^+, h^-) \geq \mathcal{E}^\mu(h^-)$. By the positive preserving property of $\mathcal{E}^\mu$

$$0 \leq \mathcal{E}^\mu(h^-) \leq \mathcal{E}^\mu(h^+, h^-) \leq 0,$$

and thus $\mathcal{E}^\mu(h^-) = 0$. Hence $\mathcal{E}(h^-) = 0$ by Lemma 3.1 and $h^- = 0$ by the transience of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. □

We showed in some paragraphs before Lemma 2.8 that the semigroup T_t^μ can be extended to a operator on $\mathcal{D}_e(\mathcal{E}^\mu)$. By the same argument for T_t^μ, the resolvent G_α^μ, $\alpha > 0$ can also be extended to a operator on $\mathcal{D}_e(\mathcal{E}^\mu)$.

Lemma 3.4. *Suppose $\lambda(\mu) > 1$. Let f and h be functions in Lemma 3.2. Then*

$$h - \alpha G_\alpha^\mu h = G_\alpha^\mu f.$$

Proof. We can construct an approximating sequence $\{h_n\} \subset \mathcal{D}(\mathcal{E}^\mu)$ of h such that $\{G_\alpha^\mu h_n\}$ is an approximation sequence of $G_\alpha^\mu h$. Hence, we have for $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$

$$\mathcal{E}_\alpha^\mu(G_\alpha^\mu h, \psi) = \lim_{n\to\infty} \mathcal{E}_\alpha^\mu(G_\alpha^\mu h_n, \psi) = \lim_{n\to\infty}(h_n, \psi) = (h, \psi)_m$$

because $\psi \in \mathcal{L}$ and

$$\int_E (h - h_n)\psi dm \le C\mathcal{E}(h - h_n)^{1/2}. \tag{3.5}$$

Hence

$$\begin{aligned}\mathcal{E}_\alpha^\mu(h - \alpha G_\alpha^\mu h, \psi) &= \int_E f\psi dm - \alpha \int_E h\psi dm + \alpha \int_E h\psi dm \\ &= \int_E f\psi dm = \mathcal{E}_\alpha^\mu(G_\alpha^\mu f, \psi),\ \forall \psi \in \mathcal{D}(\mathcal{E}) \cap C_0(E).\end{aligned}$$

□

Remark 3.5. We see from (5.1) that $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ belongs to the space $\mathcal{L}$.

Lemma 3.6. *For $f \in \mathcal{L}$, the function $G^\mu f$ belongs to $\mathcal{D}_e(\mathcal{E}^\mu)$ and equals h defined in Lemma 3.2.*

Proof. We may suppose f is non-negative, $f \ge 0$. Noting that $G_\alpha^\mu f \le h$ by Lemma 3.4, we have

$$\mathcal{E}^\mu(G_\alpha^\mu f) \le \mathcal{E}_\alpha^\mu(G_\alpha^\mu f) = \int_E f G_\alpha^\mu f dm \le \int_E h f dm < \infty,$$

and so $\sup_{\alpha>0} \mathcal{E}^\mu(G_\alpha^\mu f) < \infty$. Since $G_\alpha^\mu f \uparrow G^\mu f$ as $\alpha \to 0$, we see from Banach-Alaoglu theorem that for a certain sequence $\alpha_n \downarrow 0$, $G_{\alpha_n}^\mu f$ converges $\mathcal{E}^\mu$-weakly to $G^\mu f \in \mathcal{D}_e(\mathcal{E}^\mu)$.

Since

$$|\alpha_n (G_{\alpha_n}^\mu f, \psi)_m| \le \alpha_n (h, |\psi|)_m \to 0, \quad n \to \infty,$$

we have

$$\mathcal{E}^\mu(h, \psi) = \int_E f\psi dm = \mathcal{E}_{\alpha_n}^\mu(G_{\alpha_n}^\mu f, \psi) \to \mathcal{E}^\mu(G^\mu f, \psi), \quad n \to \infty.$$

Hence, $\mathcal{E}^\mu(h, \psi) = \mathcal{E}^\mu(G^\mu f, \psi)$ for any $\psi \in \mathcal{D}_e(\mathcal{E}^\mu)$, and $h = G^\mu f$. □

The fact in Lemma 3.6 above is proved in [3] for Dirichlet forms generated by rotationally symmetric α-stable processes.

Theorem 3.7. *If* $\lambda(\mu) > 1$, *then* $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ *is* subcritical.

Proof. For a non-negative function $f \in C_0(E)$ with $f \not\equiv 0$, $h = G^\mu f$ belongs to $\mathcal{H}^+(\mu)$ because

$$T_t^\mu G^\mu f(x) \le G^\mu f(x)$$

by lemma 2.10. Since

$$\mathcal{E}^{\mu,h}(1) = \mathcal{E}^\mu(h) = \int_E f G^\mu f dm > 0,$$

$(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is transient, more precisely, has the killing part. □

Remark 3.8. As stated in Introduction, for a general positive smooth Radon measure with $\lambda(\mu) = 1$ we cannot construct the ground state of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, and we do not know whether $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is critical or not.

4. Probabilistic representation of Schrödinger semigroups

In this section, we give a sufficient condition for μ that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable and its Schrödinger semigroup T_t^μ can be expressed by a Feynman-Kac semigroup (1.2).

Let $X = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \{P_x\}_{x\in E}, \{X_t\}_{t\ge 0}, \zeta)$ be the symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathscr{F}_t\}_{t\ge 0}$ is the augmented filtration and ζ is the lifetime of X. Denote by $\{p_t\}_{t\ge 0}$ and $\{R_\alpha\}_{\alpha\ge 0}$ the semigroup and resolvent of X:

$$p_t f(x) = E_x(f(X_t)), \qquad R_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) dt.$$

Then $p_t f(x) = T_t f(x)$ m-a.e., $R_\alpha f(x) = \int_0^\infty T_t f(x) dt$ m-a.e. Let us remember that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is supposed to be irreducible and transient throughout this paper. Consequently, the corresponding Markov process X is irreducible and transient. In the sequel, we assume that X satisfies, in addition, the next condition:

Strong Feller Property (SF). For each t, $p_t(\mathscr{B}_b(E)) \subset C_b(E)$, where $C_b(E)$ is the space of bounded continuous functions on E.

We remark that **(SF)** implies

Absolute Continuity Condition (AC). The transition probability of X is absolutely continuous with respect to m, $p(t, x, dy) = p(t, x, y)m(dy)$ for each $t > 0$ and $x \in E$.

Under **(AC)**, there exists a non-negative, jointly measurable α-resolvent kernel $R_\alpha(x,y)$: For $x \in E$ and $f \in \mathscr{B}_b(E)$

$$R_\alpha f(x) = \int_E R_\alpha(x,y) f(y) m(dy).$$

Moreover, $R_\alpha(x,y)$ is α-excessive in x and in y ([12, Lemma 4.2.4]). We simply write $R(x,y)$ for $R_0(x,y)$. For a measure μ, we define the α-potential of μ by

$$R_\alpha \mu(x) = \int_E R_\alpha(x,y) \mu(dy).$$

Let S_{00} be the set of positive Borel measures μ such that $\mu(E) < \infty$ and $R_1\mu$ is bounded. We call a Borel measure μ on E *smooth* if there exists a sequence $\{E_n\}$ of Borel sets increasing to E such that for each n $1_{E_n} \cdot \mu \in S_{00}$ and for any $x \in E$

$$P_x(\lim_{n\to\infty} \sigma_{E\setminus E_n} \geq \zeta) = 1,$$

where $\sigma_{E\setminus E_n}$ is the first hitting time of $E \setminus E_n$. We denote by S the set of smooth, positive Borel measures. In [12], a measure in S is called a *smooth measure in the strict sense*. In the sequel, we omit the adjective phrase "in the strict sense" .

Definition 4.1. Suppose that $\mu \in S$ is a positive smooth measure.

(1) μ is said to be in the *Kato class* of X ($\mathcal{K}(X)$ in abbreviation) if

$$\lim_{\alpha\to\infty} \|R_\alpha \mu\|_\infty = 0.$$

μ is said to be in the *local Kato class* ($\mathcal{K}_{loc}(X)$ in abbreviation) if for any compact set K, $1_K \cdot \mu$ belongs to $\mathcal{K}(X)$.

(2) Suppose that X is transient. A measure μ is said to be in the class $\mathcal{K}_\infty(X)$ if for any $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$

$$\sup_{x\in E} \int_{K^c} R(x,y)\mu(dy) < \epsilon.$$

μ in $\mathcal{K}_\infty(X)$ is called *Green-tight.*

A stochastic process $\{A_t\}_{t\geq 0}$ is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$.

(ii) there exists a set $\Lambda \in \mathscr{F}_\infty = \sigma(\cup_{t\geq 0}\mathscr{F}_t)$ such that $P_x(\Lambda) = 1$, for all $x \in X$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying: $A_0 = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation). The set of all PCAF's is denoted by $\mathbf{A}_c^+$. The family S and $\mathbf{A}_c^+$

are in one-to-one correspondence (*Revuz correspondence*) as follows: for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any $f \in \mathscr{B}^+(E)$ and γ-excessive function h ($\gamma \geq 0$), that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t\to 0} \frac{1}{t} E_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_E f(x)h(x)\mu(dx) \tag{4.1}$$

([12, Theorem 5.1.7]). Here, $E_{h\cdot m}(\,\cdot\,) = \int_E E_x(\,\cdot\,)h(x)m(dx)$. We denote by A_t^μ the PCAF corresponding to $\mu \in S$.

Theorem 4.2. *Let $\mu \in \mathcal{K}_{loc}(X)$. If $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite, then it is closable. Moreover, the semigroup T_t^μ generated by the closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is expressed as*

$$T_t^\mu f(x) = p_t^\mu f(x) = E_x\left(e^{A_t^\mu} f(X_t)\right) \quad m\text{-a.e.}$$

Proof. Let $\{G_n\}$ be a sequence of relatively compact open sets such that $G_n \subset \overline{G}_n \subset G_{n+1}$ and $G_n \uparrow E$. Denote by μ_n the restriction of μ to $\overline{G}_n$, $\mu_n(\cdot) = \mu(\overline{G}_n \cap \cdot)$. Then since μ_n is in the Kato class $\mathcal{K}(X)$,

$$\mathcal{E}^{\mu_n}(u) = \mathcal{E}(u) - \int_E u^2 d\mu_n, \quad u \in \mathcal{D}(\mathcal{E})$$

is a closed form on $L^2(E;m)$ and the associated $L^2(E;m)$-semigroup $\{T_t^{\mu_n}\}$ equals $\{p_t^{\mu_n}\}$ ([1, Proposition 3.1]). The sequence of closed, positive form $\{(\mathcal{E}^{\mu_n}, \mathcal{D}(\mathcal{E}))\}$ is decreasing in the sense of [24, p.373] and

$$\mathcal{E}^\mu(u) = \lim_{n\to\infty} \mathcal{E}^{\mu_n}(u), \quad u \in \mathcal{D}(\mathcal{E}).$$

Hence $(\mathcal{E}^{\mu_n}, \mathcal{D}(\mathcal{E}))$ converges to $(\mathcal{E}^*, \mathcal{D}(\mathcal{E}^*))$ in strong resolvent sense ([24, Theorem S.16]), where $(\mathcal{E}^*, \mathcal{D}(\mathcal{E}^*))$ is the closure of the largest closable form smaller than $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$.

Since $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite, in particular, lower semi-bounded, the semigroup p_t^μ is strongly continuous on $L^2(E;m)$ ([1, Theorem 4.1]).[3] For a non-negative Borel function f

$$\lim_{n\to\infty} p_t^{\mu_n} f(x) = \lim_{n\to\infty} E_x\left(e^{A_t^{\mu_n}} f(X_t)\right) = E_x\left(e^{A_t^\mu} f(X_t)\right) = p_t^\mu f(x),$$

and thus $(\mathcal{E}^*, \mathcal{D}(\mathcal{E}^*))$ is identified with the closed form $(\widetilde{\mathcal{E}}^\mu, \mathcal{D}(\widetilde{\mathcal{E}}^\mu))$ generated by $\{p_t^\mu\}$.

Define

$$\mathcal{D}(\mathcal{E}_{\overline{G}_n}) = \{u \in \mathcal{D}(\mathcal{E}) \mid u = 0 \ m\text{-a.e. on } X \setminus \overline{G}_n\}$$
$$\mathcal{D}(\widetilde{\mathcal{E}}^\mu_{\overline{G}_n}) = \{u \in \mathcal{D}(\widetilde{\mathcal{E}}^\mu) \mid u = 0 \ m\text{-a.e. on } X \setminus \overline{G}_n\}.$$

[3] In [1, Theorem 4.1], they proved the equivalence between the lower semi-boundedness of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap L^2(\mu))$ and the strong continuity of p_t^μ. By the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, the lower semi-boundedness of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap L^2(\mu))$ follows from that of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$.

We then see from [1, Theorem 5.5] that

$$\text{(i) } \mathcal{D}(\widetilde{\mathcal{E}}^{\mu}_{\overline{G}_n}) = \mathcal{D}(\widetilde{\mathcal{E}}^{\mu_n}_{\overline{G}_n}) = \mathcal{D}(\mathcal{E}_{\overline{G}_n}), \quad \text{(ii) } \widetilde{\mathcal{E}}^{\mu}(u) = \mathcal{E}^{\mu}(u),\ u \in \mathcal{D}(\mathcal{E}_{\overline{G}_n}).$$

and that the closure of $\cup_n \mathcal{D}(\widetilde{\mathcal{E}}^{\mu}_{\overline{G}_n})$ with respect to $\widetilde{\mathcal{E}}^{\mu}_1 = \widetilde{\mathcal{E}}^{\mu} + (\ ,\ )_m$ is equal to $\mathcal{D}(\widetilde{\mathcal{E}}^{\mu})$. Therefore, noting that $\mathcal{D}(\mathcal{E}) \cap C_0(E) \subset \cup_n \mathcal{D}(\widetilde{\mathcal{E}}_{\overline{G}_n})$, we can conclude that $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable and its closure $\mathcal{D}(\mathcal{E}^{\mu})$ is identified with $\mathcal{D}(\widetilde{\mathcal{E}}^{\mu})$. □

The resolvent G^{μ}_{α} is also probabilistically expressed as

$$G^{\mu}_{\alpha} f(x) = R^{\mu}_{\alpha} f(x) = E_x \left(\int_0^{\infty} e^{-\alpha t} e^{A^{\mu}_t} f(X_t) dt \right) \quad m\text{-a.e.}$$

Let D be an open set. If μ belongs to $\mathcal{K}_{\rm loc}(X^D)$, the local Kato class associated with the part process of X on D, Theorem 4.2 says that the form $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(D))$ is closable on $L^2(D;m)$. We denote $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{D,\mu}))$ its closure. Theorem 4.2 can be extended as follows:

Theorem 4.3. *Let $K \subset E$ be a compact set with* $\mathrm{Cap}(K) = 0$. *Put $D = E \backslash K$. If μ belongs to the local Kato class $\mathcal{K}_{loc}(X^D)$, then the form $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable on $L^2(E;m)$ and its closure equals $\mathcal{D}(\mathcal{E}^{D,\mu})$.*

Proof. First note that since $m(K) = 0$, the form $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(D))$ is closable on $L^2(E;m)$ and its closure equals $\mathcal{D}(\mathcal{E}^{D,\mu})$.

Take a relatively compact open set $G_1 \supset K$ and let $(\mathcal{E}^{G_1}, \mathcal{D}(\mathcal{E}^{G_1}))$ be the part Dirichlet form of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on G_1. Then $\mathrm{Cap}^{G_1}(K) = 0$ by [12, Theorem 4.4.3 (ii)], where Cap^{G_1} is the capacity defined by $(\mathcal{E}^{G_1}, \mathcal{D}(\mathcal{E}^{G_1}))$. Hence, there exists an open set G'_1 such that $K \subset G'_1 \subset G_1$ and $\mathrm{Cap}^{G_1}(G'_1) < 1$.

Next there exists a relatively compact set G_2 such that $K \subset G_2 \subset \overline{G}_2 \subset G'_1$ and the distance between K and G_2^c is less than $1/2$,

$$d(K, G_2^c) = \inf\{d(x,y) \mid x \in K,\ y \in G_2^c\} < 1/2.$$

Since $\mathrm{Cap}^{G_2}(K) = 0$, there exists an open set G'_2 such that $K \subset G'_2 \subset G_2$ and $\mathrm{Cap}^{G_2}(G'_2) < 1/2$. By repeating this procedure, we have the following sequences of open sets $\{G_n\}$, $\{G'_n\}$ such that

i) $G_1 \supset G'_1 \supset \overline{G}_2 \supset G_2 \supset G'_2 \supset \cdots \supset G_n \supset G'_n \supset \overline{G}_{n+1} \supset \cdots \supset K$,
ii) $\mathrm{Cap}^{G_n}(G'_n) < 1/n$,
iii) $d(K, G_n^c) < 1/n$.

Therefore, there exists a sequence $\{\varphi_n\}$ such that $\varphi_n \in \mathcal{D}(\mathcal{E}) \cap C_0(G_n)$, $0 \le \varphi_n \le 1$, $\varphi_n(x) = 1$ for $x \in \overline{G}_{n+1}$ and $\mathcal{E}_1(\varphi_n) < 2/n$. For any $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ define $\psi_n = \psi - \psi\varphi_n$. Then $\psi_n \in \mathcal{D}(\mathcal{E}) \cap C_0(D)$ and $\lim_{n\to\infty} \varphi_n(x) = 0$ for any $x \in D$, in particular, $\lim_{n\to\infty} \psi_n(x) = \psi$ m-a.e. Furthermore, since

$$\sup_n \mathcal{E}(\psi_n)^{1/2} \le \sup_n \left(\mathcal{E}(\psi)^{1/2} + \|\psi\|_{\infty} \mathcal{E}(\varphi_n)^{1/2} + \|\varphi_n\|_{\infty} \mathcal{E}(\psi)^{1/2} \right) < \infty,$$

there exists a subsequence of $\{\psi_n\}$ whose Cesaro mean converges to ψ with respect to $\mathcal{E}^\mu$. Hence the closure $\mathcal{D}(\mathcal{E}^{D,\mu})$ of $\mathcal{D}(\mathcal{E})\cap C_0(D)$ contains $\mathcal{D}(\mathcal{E})\cap C_0(E)$, and thus the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E})\cap C_0(E))$ is closable and the closure $\mathcal{D}(\mathcal{E}^\mu)$ equals $\mathcal{D}(\mathcal{E}^{D,\mu})$. □

5. Criticality and Hardy-type inequality

For a positive measure $\mu\in\mathcal{K}_{\rm loc}$ and a Borel set B of E, we denote by μ_B the restriction of μ to B, $\mu_B(\cdot)=\mu(B\cap\cdot)$. For $\mu\in\mathcal{K}_{\rm loc}$ define ν and ν_B by

$$\nu=\frac{\mu}{R\mu},\quad \nu_B=\frac{\mu_B}{R\mu_B}.$$

For a compact set K, the measure μ_K is in $\mathcal{K}_\infty$ and so $R\mu_K$ is bounded continuous ([7, Proposition 2.2]). Moreover, μ_K is of finite (0-order) energy integral because

$$\begin{aligned}(5.1)\qquad \int_E|\psi|d\mu_K&\le\left(\int_E\psi^2d\mu_K\right)^{1/2}\mu(K)^{1/2}\\ &\le\|R\mu_K\|_\infty^{1/2}\mu(K)^{1/2}\mathcal{E}(\psi)^{1/2},\quad \psi\in\mathcal{D}(\mathcal{E})\cap C_0(E)\end{aligned}$$

and so $R\mu_K$ belongs to $\mathcal{D}_e(\mathcal{E})$. Since

$$\begin{aligned}\mathcal{E}^{\nu_K}(R\mu_K,\varphi)&=\mathcal{E}(R\mu_K,\varphi)-\int_E R\mu_K\cdot\varphi d\nu_K\\ &=\int_E\varphi d\mu_K-\int_E\varphi d\mu_K=0,\quad \varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E),\end{aligned}$$

$R\mu_K$ is a generalized eigenfunction corresponding to the generalized eigenvalue 0. Noting that $\varphi/R\mu_K$ belongs to $\mathcal{D}(\mathcal{E})\cap C_0(E)$ by the same argument as in cite[Lemma 2.4]T4, we see from [9, Theorem 10.2] that the Schrödinger form $(\mathcal{E}^{\nu_K},\mathcal{D}(\mathcal{E}^{\nu_K}))$ is positive semi-definite because

$$\mathcal{E}^{\nu_K}(\varphi)=\mathcal{E}^{\nu_K}(R\mu_K(\varphi/R\mu_K))=\iint_{E\times E}(R\mu_K)^2d\mu_{\langle\varphi/R\mu_K\rangle}\ge 0.$$

Consequently, $(\mathcal{E}^\nu,\mathcal{D}(\mathcal{E})\cap C_0(E))$ is also positive semi-definite. In fact, for any $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$, take a compact set K such that supp$[\varphi]\subset K$

$$\begin{aligned}\mathcal{E}^\nu(\varphi)&=\mathcal{E}(\varphi)-\int_E\varphi^2\frac{d\mu}{R\mu}=\mathcal{E}(\varphi)-\int_E\varphi^2\frac{d\mu_K}{R\mu}\\ &\ge\mathcal{E}(\varphi)-\int_E\varphi^2\frac{d\mu_K}{R\mu_K}=\mathcal{E}^{\nu_K}(\varphi)\ge 0.\end{aligned}$$

Since $R\mu=\lim_{n\to\infty}R\mu_{K_n}$ for a sequence $\{K_n\}$ of compact sets increasing to E, $R\mu$ is lower semi-continuous. Since for a non-trivial smooth measure μ, $R\mu(x)>0$ by the irreducibility and so for a compact set K, $\inf_{x\in K}R\mu(x)>0$. Hence if $R\mu$ is locally bounded, then $\nu=\mu/R\mu$ is also in $\mathcal{K}_{\rm loc}$. Hence we see that for a non-trivial $\mu\in\mathcal{K}_{\rm loc}$, $(\mathcal{E}^\nu,\mathcal{D}(\mathcal{E})\cap C_0(E))$ is closable. We denote $(\mathcal{E}^\nu,\mathcal{D}(\mathcal{E}^\nu))$ its closure.

Lemma 5.1. *For a compact K and $\mu \in \mathcal{K}_{loc}$*

$$\mathcal{E}^\nu(R\mu_K) \le \iint_{K\times K^c} R(x,y)d\mu(x)d\mu(y).$$

Proof.

$$\begin{aligned}
\mathcal{E}^\nu(R\mu_K) &= \int_E R\mu_K d\mu_K - \int_E \frac{(R\mu_K)^2}{R\mu} d\mu \\
&= \int_E \left(\frac{R\mu_K(R\mu_K + R\mu_{K^c}) - (R\mu_K)^2}{R\mu_K + R\mu_{K^c}}\right) d\mu - \int_E R\mu_K d\mu_{K^c} \\
&= \int_E \frac{R\mu_K R\mu_{K^c}}{R\mu_K + R\mu_{K^c}} d\mu - \int_E R\mu_K d\mu_{K^c} \\
&= \int_E \frac{R\mu_K R\mu_{K^c}}{R\mu_K + R\mu_{K^c}} d\mu_K + \int_E \frac{R\mu_K R\mu_{K^c}}{R\mu_K + R\mu_{K^c}} d\mu_{K^c} - \int_E R\mu_K d\mu_{K^c}.
\end{aligned}$$

Since

$$\frac{R\mu_K R\mu_{K^c}}{R\mu_K + R\mu_{K^c}} \le R\mu_K, \quad \frac{R\mu_K R\mu_{K^c}}{R\mu_K + R\mu_{K^c}} \le R\mu_{K^c},$$

the right hand side is less than or equal to

$$\int_E R\mu_{K^c} d\mu_K + \int_E R\mu_K d\mu_{K^c} - \int_E R\mu_K d\mu_{K^c}.$$

Noting

$$\int_E R\mu_{K^c} d\mu_K = \int_E R\mu_K d\mu_{K^c} = \iint_{K\times K^c} R(x,y)d\mu(x)d\mu(y),$$

we have the lemma. □

We define a subclass $\mathcal{K}_H$ of $\mathcal{K}_{\rm loc}$ as follows: a measure $\mu \in \mathcal{K}_{\rm loc}$ belongs to $\mu \in \mathcal{K}_H$ if μ satisfies that $R\mu$ is in $\mathcal{D}_{\rm loc}(\mathcal{E}) \cap \mathscr{B}_{b,{\rm loc}}(E)$ and there exists an increasing sequence $\{K_n\}$ of compact sets such that $K_n \uparrow E$ and

$$\sup_n \iint_{K_n\times K_n^c} R(x,y)d\mu(x)d\mu(y) < \infty. \tag{5.2}$$

Lemma 5.2. *If $\mu \in \mathcal{K}_H$, then $R\mu$ belongs to $\mathcal{E}_e(\mathcal{E}^\nu)$.*

Proof. Let $\{K_n\}$ be a sequence of compact sets for μ in (5.2). Then by Lemma 5.1

$$\sup_n \mathcal{E}^\nu(R\mu_{K_n}) < \infty.$$

Since $\lim_{n\to\infty} R\mu_{K_n} = R\mu$, $R\mu \in \mathcal{D}_e(\mathcal{E}^\nu)$. □

Lemma 5.3. *Let $\mu \in \mathcal{K}_H$. For $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$*

$$\mathcal{E}^\nu(R\mu, \varphi) = 0.$$

Proof. Since $\sup_n \mathcal{E}^\nu(R\mu_{K_n}) < \infty$, there exists a subsequence $\{K_{n_l}\} \subset \{K_n\}$ such that

$$R\left(\frac{(1_{K_{n_1}} + 1_{K_{n_2}} \cdots + 1_{K_{n_l}})}{l}\mu\right) \longrightarrow R\mu$$

with $\mathcal{E}^\nu$-strongly. Let $0 \le \phi_l := (1_{K_{n_1}} + 1_{K_{n_2}} \cdots + 1_{K_{n_l}})/l \le 1$. Then $\phi_l \to 1$.

For a fixed $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ we can assume $\mathrm{supp}[\varphi] \subset K_{n_1}$. Then

$$\mathcal{E}^\nu(R\mu, \varphi) = \lim_{l\to\infty} \mathcal{E}^\nu(R(\phi_l\mu), \varphi) = \lim_{l\to\infty}\left(\mathcal{E}(R(\phi_l\mu), \varphi) - \int_E \frac{R(\phi_l\mu)}{R\mu}\varphi d\mu\right).$$

Note that $R(\phi_l\mu)$ belongs to $\mathcal{D}_e(\mathcal{E})$. Then since

$$\lim_{l\to\infty} \mathcal{E}(R(\phi_l\mu), \varphi) = \lim_{l\to\infty} \int_E \phi_l \varphi d\mu = \int_E \varphi d\mu$$

and by the monotone convergence theorem

$$\lim_{l\to\infty} \int_E \frac{R(\phi_l\mu)}{R\mu}\varphi d\mu = \int_E \varphi d\mu,$$

we have the lemma. □

Theorem 5.4. *If $\mu \in \mathcal{K}_H$, then $R\mu$ is a ground state of $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$, consequently, $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ is critical.*

Proof. Since $R\mu$ belongs to $\mathcal{D}_e(\mathcal{E}^\nu)$, there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that φ_n converges to $R\mu$ $\mathcal{E}^\nu$-strongly. Hence

$$\mathcal{E}^\nu(R\mu) = \lim_{n\to\infty} \mathcal{E}^\nu(R\mu, \varphi_n) = 0.$$

□

Example 5.5. Let X be a one-dimensional diffusion process on $(0, \infty)$ with scale function S such that $\lim_{x\to 0} S(x) = 0$, $\lim_{x\to\infty} S(x) = \infty$. Then 0-resolvent kernel is given by

$$R(x, y) = S(x) \wedge S(y).$$

Let $\mu \in \mathcal{K}_{\mathrm{loc}}$ satisfying

$$\sup_{r>0}\left(\mu((r,\infty))\int_0^r S d\mu\right) < \infty. \tag{5.3}$$

We then have a optimal Hardy inequality

$$\int_0^\infty u^2 \frac{1}{R\mu} d\mu \le \int_0^\infty \left(\frac{du}{dS}\right)^2 dS. \tag{5.4}$$

If $S(x) = x^p$, $p > 0$, then $\mu(dx) = x^{-(p+2)/2}dx$ satisfies (5.3) and the equation (5.4) gives us the inequality

$$\int_0^\infty \left(x^{-(p+1)/2} \cdot u\right)^2 dx \le \frac{4}{p^2} \int_0^\infty \left(x^{-(p-1)/2} \cdot \frac{du}{dx}\right)^2 dx, \tag{5.5}$$

which is the classical one when $p = 1$.

Example 5.6. Let us consider the symmetric α-stable process $X^{(\alpha)}$ on $\mathbb{R}^d$ with $0 < \alpha < 2$. We assume that $\alpha < d$, that is, $X^{(\alpha)}$ is transient. Let $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ be the Dirichlet form generated by $X^{(\alpha)}$:

$$\begin{cases} \mathcal{E}^{(\alpha)}(u) = \dfrac{1}{2}\mathcal{A}(d,\alpha) \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \dfrac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dxdy \\ \mathcal{D}(\mathcal{E}^{(\alpha)}) = \left\{ u \in L^2(\mathbb{R}^d) \mid \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \dfrac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dxdy < \infty \right\}, \end{cases} \tag{5.6}$$

where $\triangle = \{(x,x) \mid x \in \mathbb{R}^d\}$ and

$$\mathcal{A}(d,\alpha) = \frac{\alpha 2^{d-1}\Gamma(\frac{\alpha+d}{2})}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$$

([12, Example 1.4.1]).

Let $R(x,y)$ be the 0-resolvent density of the rotational symmetric α-stable process X on $\mathbb{R}^d$, that is,

$$R(x,y) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2}\Gamma(\alpha/2)} \cdot \frac{1}{|x-y|^{d-\alpha}}.$$

We see from Hardy-Littlewood-Sobolev inequality ([18, Theorem 4.3]) that for $1 < p < q$, $1/p + 1/q = (d+\alpha)/d$, and $d/q < \beta < d/p$

$$\begin{aligned} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1_{B(r)}(x)1_{B(r)^c}(y)}{|x-y|^{d-\alpha}|x|^\beta|y|^\beta} dxdy &\le C \cdot \|1_{B(r)}|x|^{-\beta}\|_p \|1_{B(r)^c}|x|^{-\beta}\|_q \\ &= C \cdot r^{d/p-\beta} r^{d/q-\beta} \\ &= C \cdot r^{d(1/p+1/q)-2\beta}, \end{aligned}$$

where $B(r) = \{x \in \mathbb{R}^d \mid |x| \le r\}$. Hence if

$$d\left(\frac{1}{p} + \frac{1}{q}\right) - 2\beta = 0 \iff \beta = \frac{d+\alpha}{2},$$

then

$$M := \sup_{r>0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1_{B(r)}(x)1_{B(r)^c}(y)}{|x-y|^{d-\alpha}|x|^\beta|y|^\beta} dxdy < \infty.$$

Note that there exists $1 < p < q$ such that $1/p + 1/q = (d+\alpha)/d$ and $d/q < (d+\alpha)/2 < d/p$.

Let X^D be the part process of X on $D = \mathbb{R}^d \setminus \{0\}$. Then the measure $\mu(dx) = |x|^{-(d+\alpha)/2}dx$ belongs to $\mathcal{K}_H(X^D)$. Indeed, μ is in $\mathcal{K}_{\mathrm{loc}}(X^D)$. for

$T_n = \{x \in \mathbb{R}^d \mid 1/n \le |x| \le n\}$

$$\begin{aligned}&\sup_n \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1_{T_n}(x)1_{T_n^c}(y)}{|x-y|^{d-\alpha}|x|^{(d+\alpha)/2}|y|^{(d+\alpha)/2}}dxdy\\ \le &\sup_n \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1_{B(n)}(x)1_{B(n)^c}(y)+1_{B(1/n)}(x)1_{B(1/n)^c}(y)}{|x-y|^{d-\alpha}|x|^{(d+\alpha)/2}|y|^{(d+\alpha)/2}}dxdy\\ \le & 2M<\infty.\end{aligned}$$

The capacity of $\{0\}$ is zero with respect to the α-stable process X and thus the closure of $\mathcal{D}(\mathcal{E}^{(\alpha)})\cap C_0(D)$ and that of $\mathcal{D}(\mathcal{E}^{(\alpha)})\cap C_0(\mathbb{R}^d)$ is equal. Note that

$$R\mu(x) = \frac{\Gamma((d-\alpha)/4)^2}{2^\alpha\Gamma((d+\alpha)/4)^2}\cdot|x|^{-(d-\alpha)/2}.$$

and so

$$\frac{\mu(dx)}{R\mu(x)} = \kappa^*\frac{1}{|x|^\alpha}, \quad \kappa^* = \frac{2^\alpha\Gamma((d+\alpha)/4)^2}{\Gamma((d-\alpha)/4)^2}.$$

Then, by Theorem 5.4, $R\mu$ is a ground state of

$$\mathcal{E}^\nu(u) = \mathcal{E}^{(\alpha)}(u) - \kappa^*\int_{\mathbb{R}^d} u^2/|x|^\alpha dx$$

and $(\mathcal{E}^\nu, \mathcal{D}(\mathcal{E}^\nu))$ is critical. We see that the constant κ^* equals the best constant of Hardy inequality.

Let $\mu \in \mathcal{K}_H$. By Revuz correspondence, $R\mu(x)$ is written as $E_x(A^\mu_\zeta)$ and

$$\begin{aligned}R\mu(X_t) &= E_{X_t}(A^\mu_\zeta) = E_x(A^\mu_\zeta(\theta_t)1_{\{t<\zeta\}}|\mathscr{F}_t)\\ &= E_x((A^\mu_\zeta - A^\mu_t)1_{\{t<\zeta\}}|\mathscr{F}_t)\\ &= E_x(A^\mu_\zeta|\mathscr{F}_t) - A^\mu_t, \quad t<\zeta \quad P_x\text{-a.s.}\end{aligned}$$

Define a multiplicative functional L_t by

$$L_t = \frac{R\mu(X_t)}{R\mu(X_0)}\exp\left(\int_0^t \frac{dA^\mu_s}{R\mu(X_s)}\right).$$

Put

$$M_t = E_x(A^\mu_\zeta|\mathscr{F}_t), \ V_t = \exp\left(\int_0^t \frac{dA^\mu_s}{R\mu(X_s)}\right).$$

Then $R\mu(X_t) = M_t - A^\mu_t$, $t<\zeta$ and by Itô formula

$$\begin{aligned}L_t &= 1 + \frac{1}{R\mu(X_0)}\left(\int_0^t V_s dR\mu(X_s) + \int_0^t R\mu(X_s)V_s\frac{dA^\mu_s}{R\mu(X_s)}\right)\\ &= 1+\int_0^t V_s dM_s, \quad t<\zeta.\end{aligned}$$

Suppose the Hunt process X has no killing inside. Then the life time ζ is predictable and $\int_0^t V_s dM_s$ becomes a local martingale. Hence

$$E_x(L_t) = \frac{1}{R\mu(x)} E_x\left(\exp\left(\int_0^t \frac{dA_s^\mu}{R\mu(X_s)}\right) R\mu(X_t)\right) \leq 1.$$

In other words, $R\mu$ is p_t^ν-excessive, $p_t^\nu R\mu \leq R\mu$. Hence we have the next corollary.

Corollary 5.7. *For $\mu \in \mathcal{K}_H$ let $\nu = \mu/R\mu$ and define*

$$\begin{cases} \mathcal{E}^{\nu,R\mu}(u,u) = \mathcal{E}^\nu(R\mu \cdot u, R\mu \cdot u) \\ \mathcal{D}(\mathcal{E}^{\nu,R\mu}) = \{u \in L^2(E;(R\mu)^2 m) \mid R\mu \cdot u \in \mathcal{D}(\mathcal{E}^\nu)\}. \end{cases} \tag{5.7}$$

Then $(\mathcal{E}^{\nu,R\mu}, \mathcal{D}(\mathcal{E}^{\nu,R\mu}))$ is a recurrent Dirichlet form.

6. Symmetric α-stable process: Recurrence and transience

In this section, we apply the results obtained in previous section to Fractional Schrödinger operators with Hardy potential.

Example 6.1. Let us consider the symmetric α-stable process $X^{(\alpha)}$ on $\mathbb{R}^d$ with $0 < \alpha < 2$ and $\alpha < d$. For $0 \leq \delta \leq d - \alpha$, define a smooth Radon measure μ^δ by

$$\mu^\delta = \kappa(\delta)|x|^{-\alpha}dx,$$

where $\kappa(\delta)$ is the constant in (1.14). Let

$$\delta^* = \frac{d-\alpha}{2}, \qquad \kappa^* = \kappa(\delta^*).$$

Then κ^* is the best constant in the hardy inequality for $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ given by (1.15) and μ^{δ^*} equals ν in Example 5.6. It is known in [4]) that

$$\kappa(\delta) < \kappa^* \text{ for } \delta \neq \delta^*, \qquad \kappa(\hat{\delta}) = \kappa(\delta) \text{ for } \hat{\delta} = d - \alpha - \delta.$$

It follows from these facts that

$$\delta \neq \delta^* \iff \lambda(\mu^\delta) > 1. \tag{6.1}$$

By Theorem 2.11 and Lemma 2.10, if $\delta \neq \delta^*$, then $(\mathcal{E}^{\mu^\delta}(:= \mathcal{E}^{(\alpha),\mu^\delta}), \mathcal{D}(\mathcal{E}^{\mu^\delta}))$ is subcritical and $(\mathcal{E}^{\mu^\delta,h}, \mathcal{D}(\mathcal{E}^{\mu^\delta,h}))$ is transient for any $h \in \mathcal{H}^+(\mu^\delta)$. We know from [4, Theorem 3.1] that for $0 \leq \delta \leq \delta^*$

$$p_t^{\mu^\delta}(|x|^{-\delta}) = |x|^{-\delta}, \tag{6.2}$$

in particular, the function $|x|^{-\delta}$ belongs to $\mathcal{H}^+(\mu^\delta)$.

Let us simply denote $\mathcal{E}^\delta$ for $\mathcal{E}^{\mu^\delta,|x|^{-\delta}}$. It is shown in [4, Theorem 5.4] that $\mathcal{E}^{\delta,|x|^{-\delta}}$ is expressed as

$$\mathcal{E}^\delta(u) = \frac{1}{2}\mathcal{A}(d,\alpha)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}|x|^\delta|y|^\delta}dxdy,\ u \in C_0^\infty(\mathbb{R}^d) \tag{6.3}$$

and the closures of $(\mathcal{E}^\delta, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d; |x|^{-2\delta}dx)$ is identified with $\mathcal{D}(\mathcal{E}^\delta)$. Moreover, It is shown in [4, Theorem 5.4] that the closure of $(\mathcal{E}^\delta, C_0^\infty(\mathbb{R}^d\backslash\{0\}))$ also is identified with $\mathcal{D}(\mathcal{E}^\delta)$, which implies the capacity of $\{0\}$ with respect to $(\mathcal{E}^\delta, \mathcal{D}(\mathcal{E}^\delta))$ equals zero.

As a result, we see that for $0 \le \delta < \delta^*$, the Dirichlet form $(\mathcal{E}^\delta, \mathcal{D}(\mathcal{E}^\delta))$ is transient, which leads us to the transience of $(\mathcal{E}^\delta, \mathcal{D}(\mathcal{E}^\delta))$ for $\delta^* < \delta \le d-\alpha$ (Remark 6.3 below). Note that (6.2) implies the Markov process generated by $(\mathcal{E}^\delta, \mathcal{D}(\mathcal{E}^\delta))$ is conservative.

If $\delta = \delta^*$, then $(\mathcal{E}^{\mu^{\delta^*}}, \mathcal{D}(\mathcal{E}^{\mu^{\delta^*}}))$ is critical and $(\mathcal{E}^{\delta^*}, \mathcal{D}(\mathcal{E}^{\delta^*}))$ is recurrent ([21]). Here we give an elementary proof by making an approximation sequence of the identity function 1. First note that $C_0^{\mathsf{lip}}(\mathbb{R}^d\backslash\{0\}))$ is included in $\mathcal{D}(\mathcal{E}^{\delta,|x|^{-\delta}})$, where $C_0^{\mathsf{lip}}(\mathbb{R}^d\backslash\{0\}))$ is the set of all Lipschitz continuous functions compactly supported in $\mathbb{R}^d\backslash\{0\}$. Let f_n be the function on $[0,\infty)$ such that

$$f_n(t) = \begin{cases} 1, & 0 \le t \le n, \\ 2n-t, & n \le t \le 2n, \\ 0, & t \ge 2n \end{cases}$$

and put $\phi_n(x) = f_n(|x|)$. We define the function $\varphi_n(x)$ by

$$\varphi_n(t) = \begin{cases} \phi_n(x), & |x| \ge 1, \\ \phi_n(Tx), & 0 < |x| \le 1, \end{cases} \tag{6.4}$$

where T is a map from $\mathbb{R}^d\backslash\{0\}$ to $\mathbb{R}^d\backslash\{0\}$ defined by

$$Tx = x/|x|^2. \tag{6.5}$$

Then $\varphi_n \in C_0^{\mathsf{lip}}(\mathbb{R}^d\backslash\{0\})$ for each n.

Since $\varphi_n(x) \uparrow 1$ and $(\mathcal{E}^{\delta,|x|^{-\delta}}, \mathcal{D}(\mathcal{E}^{\delta,|x|^{-\delta}}))$ is the Dirichlet form of pure-jump type, it is enough to see the following estimate in order to obtain the recurrence of the form (see [34]):

$$\sup_n \iint_{\mathbb{R}^d\times\mathbb{R}^d\backslash\triangle} \frac{(\varphi_n(x)-\varphi_n(y))^2}{|x-y|^{d+\alpha}|x|^{(d-\alpha)/2}|y|^{(d-\alpha)/2}} dxdy < \infty. \tag{6.6}$$

Put

$$\Phi_n(x,y) = \frac{(\varphi_n(x)-\varphi_n(y))^2}{|x-y|^{d+\alpha}|x|^{(d-\alpha)/2}|y|^{(d-\alpha)/2}} \quad \text{for } (x,y) \in \mathbb{R}^d\times\mathbb{R}^d \text{ with } x \ne y.$$

The map T satisfies

$$|Tx-Ty|^{d+\alpha}|Tx|^\delta|Ty|^\delta = |x-y|^{d+\alpha}|x|^{-d-\alpha-\delta}|y|^{-d-\alpha-\delta},$$

and the Jacobian of T equals $1/|x|^{2d}$. As a result, we see from the definition of φ_n that

$$\iint_{\{|x|\le 1\}\times\{|y|\le 1\}} \Phi_n(x,y)dxdy = \iint_{\{|x|\ge 1\}\times\{|y|\ge 1\}} \Phi_n(x,y)dxdy,$$

and so

$$\begin{aligned}&\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Phi_n(x,y)dxdy\\&= 2\left(\iint_{\{|x|\le 1\}\times\{|y|\ge 1\}}\Phi_n(x,y)dxdy+\iint_{\{|x|\ge 1\}\times\{|y|\ge 1\}}\Phi_n(x,y)dxdy\right)\\&=: 2(\mathrm{I}+\mathrm{II}).\end{aligned}$$

Since $\Phi_n(x,y)=0$ for $(x,y)\in\{1/n\le|x|\le 1\}\times\{1\le|y|\le n\}$, the integral in I can be decomposed into two parts as follows:

$$\begin{aligned}\mathrm{I}&=\iint_{\{|x|\le 1/n\}\times\{|y|\ge 1\}}\Phi_n(x,y)dxdy+\iint_{\{1/n\le|x|\le 1\}\times\{|y|\ge n\}}\Phi_n(x,y)dxdy\\&=: \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

The inequality $|x-y|\ge(1-|x|/|y|)\,|y|\ge(1-1/n)\,|y|$ holds for

$$(x,y)\in\Big(\{|x|\le 1/n\}\times\{|y|\ge 1\}\Big)\bigcup\Big(\{1/n\le|x|\le 1\}\times\{|y|\ge n\}\Big).$$

So we have $\Phi_n(x,y)\le c|x|^{-(d-\alpha)/2}|y|^{-(3d+\alpha)/2}$ and I_1 and I_2 converges to 0 as $n\to\infty$.

Noting that Φ_n is a symmetric function on $\mathbb{R}^d\times\mathbb{R}^d$ and vanishes on the set $\{|x|\ge 3n\}\times\{|y|\ge 3n\}$, divide the integral in (II) into two parts:

$$\begin{aligned}\mathrm{II}&=2\iint\limits_{\{1\le|x|\le 3n\}\times\{3n\le|y|\}}\Phi_n(x,y)dxdy+2\iint\limits_{\{1\le|x|\le|y|\le 3n\}}\Phi_n(x,y)dxdy\\&=: 2(\mathrm{II}_1+\mathrm{II}_2).\end{aligned}$$

Since φ_n vanishes on the set $\{2n\le|x|\}$ and the estimates $|x-y|\ge(1/3)|y|$ hold for $(x,y)\in\{1\le|x|\le 2n\}\times\{|y|\ge 3n\}$, we see that

$$\mathrm{II}_1\le c\iint\limits_{\{1\le|x|\le 2n\}\times\{|y|\ge 3n\}}|x|^{-(d-\alpha)/2}|y|^{-(3d+\alpha)/2}dxdy\le C.$$

The Lipschitz continuity $|\varphi_n(x)-\varphi_n(y)|\le(1/n^2)|x-y|$ tells us

$$\begin{aligned}\mathrm{II}_2&\le c/n^2\iint\limits_{\{1\le|x|\le|y|\le 3n\}}|x|^{-(d-\alpha)/2}|y|^{-(d-\alpha)/2}|x-y|^{-(d+\alpha-2)}dxdy\\&\le c/n^2\int_{\{1\le|x|\le 3n\}}|x|^{-(d-\alpha)}dx\int_{\{|y|\le 6n\}}|y|^{-(d+\alpha-2)}dy\le C.\end{aligned}$$

Therefore we have the estimate (6.6). Thus we find that $(\mathcal{E}^{\delta^*},\mathcal{D}(\mathcal{E}^{\delta^*}))$ is recurrent.

Remark 6.2. Suppose $\delta\ne\delta^*$. By the Sobolev inequality,

$$L^{(2d)/(d+\alpha)}\subset\mathcal{L}.$$

Hence by Lemma 3.6 $G^\mu f \in \mathcal{D}_e(\mathcal{E}^\mu)$ for any $f \in L^{(2d)/(d+\alpha)}$.

Remark 6.3. We know form Example 6.1 that

$$\sup_n \mathcal{E}^{\mu^{\delta^*}}(|x|^{-\delta^*}\varphi_n) = \sup_n \mathcal{E}^{\delta^*}(\varphi_n) < \infty,$$

and $|x|^{-\delta^*}\varphi_n(x) \uparrow |x|^{-\delta^*}$ as $n \to \infty$ for $x \neq 0$. Hence, the function $|x|^{-\delta^*}$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu^{\delta^*}})$ and $\mathcal{E}^{\mu^{\delta^*}}(|x|^{-\delta^*}) = 0$, that is, it is the ground state. Therefore, we see that $(\mathcal{E}^{\mu^{\delta^*}}, \mathcal{D}(\mathcal{E}^{\mu^{\delta^*}}))$ is critical in the sense of Definition 2.2 (2).

Since

$$\int_{\mathbb{R}^d} |x|^{-2\delta^*}|x|^{-\alpha}dx = \int_{\mathbb{R}^d} |x|^{-d}dx = \infty,$$

the function $|x|^{-\delta^*}$ does not belong to $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ and thus $\mathcal{D}_e(\mathcal{E}^{(\alpha)}) \subset \mathcal{D}_e(\mathcal{E}^{\mu^{\delta^*}})$. We see that the minimizer of $\lambda(\mu^{\delta^*})(=1)$ does not exist, in particular, $(\mathcal{D}_e(\mathcal{E}^{(\alpha)}), \mathcal{E}^{(\alpha)})$ is not embedded in $L^2(\mathbb{R}^d, |x|^{-\alpha}dx)$.

Remark 6.4. Denote by X^δ the Markov process generated by the regular Dirichlet form $(\mathcal{E}^\delta, \mathcal{D}(\mathcal{E}^\delta))$ on $L^2(\mathbb{R}^d, 1/|x|^{2\delta}dx)$. Let T be the map in (6.5). Then the transformed process $T(X_t^\delta)$ is symmetric with respect to $1/(|x|^{2d-2\delta})dx$ and the Dirichlet form generated by $T(X_t^\delta)$ is written as

$$\frac{1}{2}\mathcal{A}(d,\alpha)\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d-\alpha}|x|^{d-\alpha-\delta}|y|^{d-\alpha-\delta}}dxdy,$$

that is, the Dirichlet form $(\mathcal{E}^{\hat{\delta}}, \mathcal{D}(\mathcal{E}^{\hat{\delta}}))$ on $L^2(\mathbb{R}^d, 1/|x|^{2d-2\delta}dx)$. Noting $2d - 2\delta = 2\hat{\delta} + 2\alpha$, we see that the time-changed process of $X_t^{\hat{\delta}}$ by $\int_0^t 1/|X_s^{\hat{\delta}}|^{2\alpha}ds$ is identified with $T(X_t^\delta)$. Let $\tau_t = \inf\{s > 0 \mid \int_0^s (1/|X_u^{\hat{\delta}}|^{2\alpha}du > t\}$. Then $X_{\tau_t}^{\hat{\delta}}$ has the same law as that of $T(X_t^\delta)$. In other words, X_t^δ has the inversion property (cf, [2]). By the time-change, the recurrence and transience are invariant, and thus the transience of X^δ implies that of $X^{\hat{\delta}}$.

References

[1] S. Albeverio, Z-M. Ma : *Perturbation of Dirichlet forms–lower semiboundedness, closability, and form cores*, J. Funct. Anal. **9**9 (1991), 332–356.

[2] L. Alili, L. Chaumont, P. Graczyk, T. Zak : *Space and time inversions of stochastic processes and Kelvin transform*, Math. Nachr. **2**92 (2019), 252–272.

[3] M. Bhakta, A. Biswas, D. Ganguly, L. Montoro : *Integral representation of solutions using Green function for fractional Hardy equations*, J. Differential Equations **269** (2020), 5573–5594.

[4] K. Bogdan, G. Krzysztof, T. Jakubowski, D. Pilarczyk : *Fractional Laplacian with Hardy potential*, Comm. Partial Differential Equations **4**4 (2019), 20–50.

[5] K. Bogdan, B. Dyda, P. Kim: Hardy inequalities and non-explosion results for semigroups, Potential Anal. **4**4 (2016), 229–2477.

[6] K. Bogdan, T. Zak: On Kelvin transformation, J. Theoret. Probab. **1**9 (2006), 89–120.

[7] Z.-Q. Chen : *Gaugeability and conditional gaugeability*, Trans. Amer. Math. Soc. **354** (2002), 4639–4679.

[8] E.B. Davies : Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge (1989).

[9] R.L. Frank, D. Lenz, D. Wingert : *Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory*, J. Funct. Anal. **2**66 (2014), 4765–4808.

[10] P.J. Fitzsimmons : *Hardy's inequality for Dirichlet forms*, J. Math. Anal. Appl. **2**50 (2000), 548–560.

[11] P. J. Fitzsimmons, K. Kuwae : *Non-symmetric perturbations of symmetric Dirichlet forms*, J. Funct. Anal. **2**08 (2004), 140–162.

[12] M. Fukushima, Y. Oshima and M. Takeda : *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, 2nd ed. (2011).

[13] B. Jaye, V.G. Maz'ya, I.E. Verbitsky : *Existence and regularity of positive solutions of elliptic equations of Schrödinger type*, J. Anal. Math. **1**18 (2012), 577–621.

[14] N. Kajino : *Equivalence of recurrence and Liouville property for symmetric Dirichlet forms*, Mat. Fiz. Komp'yut. Model (2017), 89–98.

[15] M. Keller, Y. Pinchover and F. Pogorzelski : *Criticality theory for Schrödinger operators on graphs*, J. Spectr. Theory **1**0 (2020), 73–114.

[16] D. Lenz, P. Stollmann and I. Veselić : *The Allegretto-Piepenbrink theorem for strongly local Dirichlet forms*, Doc. Math. **1**4 (2009), 167–189.

[17] L. Li : *Criticality and subcriticality of generalized Schrödinger forms with non-local perturbations*, Proc. Amer. Math. Soc. **1**45 (2017), 3929–3939.

[18] Lieb, E.H., Loss, M.: Analysis, Graduate Studies in Mathematics **14**, American Mathematical Society (2001).

[19] M. Lucia, S. Prashanth ; *Criticality theory for Schrödinger operators with singular potential*, J. Differential Equations **2**65 (2018), 3400–3440.

[20] Y. Miura : *Superharmonic functions of Schrödinger operators and Hardy inequalities*, J. Math. Soc. Japan **7**1 (2019), 689–708.

[21] Y. Miura : *Optimal Hardy inequalities for Schrödinger operators based on symmetric stable processes*, preprint.

[22] B, Devyver, M. Fraas, Y. Pinchover : Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon, J. Funct. Anal. **2**66 (2014), 4422–4489.

[23] M. Keller, Y. Pinchover and F. Pogorzelski : *Criticality theory for Schrödinger operators on graphs*, J. Spectr. Theory **1**0 (2020), 73–114.

[24] M. Reed and B. Simon : *Methods of modern mathematical physics, I. Functional analysis*, Academic Press, 2nd ed. (1980).

[25] B. Schmuland : *Extended Dirichlet spaces*, C. R. Math. Acad. Sci. Soc. R. Can. **2**1 (1999), 146–152.

[26] B. Schmuland : *Positivity preserving forms have the Fatou property*, Potential Anal. **1**0 (1999), 373–378.

[27] P. Stollmann and J. Voigt : *Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), 109–138.

[28] M. Takeda : *Criticality and subcriticality of generalized Schrödinger forms*, Illinois J. Math. **5**8 (2014), 251–277.

[29] M. Takeda : *The bottom of the spectrum of time-changed processes and the maximum principle of Schrödinger operators*, J. Theoret. Probab. **3**1 (2018), 741–756.

[30] M. Takeda : *Compactness of symmetric Markov semigroups and boundedness of eigenfunctions*, Trans. Amer. Math. Soc. **3**72 (2019), 3905–3920.

[31] M. Takeda : *Maximum principles for generalized Schrödinger equations*, Illinois J. Math. **6**4 (2020), 119–139.

[32] M. Takeda : *Criticality of Schrödinger forms and Liouville-type property*, to appear in Math. Nachr.

[33] Takeda, M., Tsuchida, K.: Differentiability of spectral functions for symmetric α-stable processes, Trans. Amer. Math. Soc. **3**59, 4031-4054 (2007).

[34] T. Uemura : *On some path properties of symmetric stable-like processes for one dimension*, Potential Analysis, **16** (2002), 79–91

Department of Mathematics, Kansai University, Yamatecho, Suita, 564-8680, Japan

E-mail address: mtakeda@kansai-u.ac.jp

Department of Mathematics, Kansai University, Yamatecho, Suita, 564-8680, Japan

E-mail address: t-uemura@kansai-u.ac.jp