The Homotopy Continuation Method and the Walrasian General Equilibrium Theory

Takashi SHIOMURA *

Abstract

The homotopy continuation method becomes a powerful tool for comparative statics in the large. It is also, by its nature, to be used for showing the existence of zeroes of a map. In this paper, we first present some existence theorems concerning set-valued maps. These are obtained from a fundamental theorem on a homotopy continuation method and theorems on continuos selections. We then apply them to the Walrasian general equilibrium models and show the existence of equilibria. Subsequently, we consider a regular economy, and get some insights into the number of equilibria due to the homotopy invariance theorem and the norm-coerciveness theorem. Although these problems have been investigated by many economists, the homotopy continuation method clarifies in a clear and systematic manner why our results have been obtained.

*The paper is a part of Shiomura (1996). I would like to express my appreciation for the financial support I have recieved from the Faculty of Informatics, Kansai University.

1 Fixed-point algorithms and economic applications

By the nature of a fixed point algorithm, it is closely related to an existence problem of economic equilibria. Perhaps the simplest but most powerful fixed point theorem which can be computationally implemented is that of Banach, i.e., the contraction mapping theorem. It can maintain uniqueness and existence of a fixed point at one stroke. It can be used also to show the convergence of the Picard iteration, so that it may be used to investigate the stability of an equilibrium.

In fact, some authors applied the theorem or its variants to economic analysis; e.g., Gale (1964) to a nonlinear Leontief input-output model; Hadar (1965-6) to Warlasian and Cournot models; Okuguchi (1970), and Okuguchi and Szidarovszky (1990) to oligopoly models; Fujimoto (1988, 1990) to Walrasian models. Amongst others, Thorlund-Petersen (1985) provided a convergence theorem, a generalization of the simple contraction mapping theorem, and examined global stability of equilibria in several economic models.

On the other hand, Uzawa (1960) showed the global stability of an equilibrium of the successive tâtonnement process, while Gabay and Moulin (1980) and Okuguchi and Szidarovszky (1990) studied the global stability of an Cournot equilibrium of a sequential adjustment process. The process considered by these authors is, indeed, the Gauss-Seidel process.

After Scarf (1967) provided a new constructive method for finding equilibria, various fixed point algorithms were used to show the existence of a Walrasian competitive equilibrium; e.g., Scarf (1973) with the collaboration of Hansen, Todd (1976), and Doup et al. (1987). All these works use simplicial algorithms.

The complementarity problem which is closely related to the fixed point problem is also discussed in economic analyses. In fact, both problems have the same origin, namely, the linear programming problem.

Eaves (1976) and Mayerson (1981) discussed the existence of equilibria for linear economic models. The former investigated a linear pure exchange economy by using Lemke's linear complementary pivoting algorithm. The latter studied a linear monetary economy and showed that the equilibrium can be found by solving a finite sequence of linear programming problems. Okuguchi (1983), and Okuguchi and Szidarovszky (1990) argued that a competitive equilibrium can be obtained by solving a nonlinear complementarity problem and, in particular, the latter proposed an algorithm which enables us to find a Cournot-Nash equilibrium for linear oligopoly models.

On the other hand, Mathiesen (1985) showed that a competitive equilibrium can be computed by solving a sequence of linear complementarity problems. Interestingly enough, Preckel (1985) examined several alternative solution methods including sequences of linear programs, PL homotopy methods, quasi-Newton methods, and sequences of linear complementarity problems, and indicated that computing equilibria by sequences of linear complementarity problems is quite efficient for large-scale problems.

The homotopy methods were also used for showing economic equilibria. Along the lines of Kellogg et al. (1976), who were early researchers into the homotopy methods, Smale (1976) proved, under some rather strong boundary conditions, the existence of a Walrasian equilibrium, using an algorithm known as the global Newton method. Kamiya (1990) also considered an algorithm which is a mixture of Walras' tâtonnement and the global Newton method.

Zangwill and Garcia (1981a-b) used an interesting homotopy continuation method for obtaining economic equilibria. In order to find a Walrasian equilibrium, they first consider a so-called abstract economy. Then, the system of inequalities, which consists of the optimal conditions of all players, are modified into the system of equations. Finally, the system is deformed until we reach an equilibrium. A similar argument is applied to a Cournot-Nash equilibrium.

Algorithms for computing solutions to systems of nonlinear equations with a specific structure were provided by Mansur and Whalley (1982), van der Laan (1985) and Kamiya (1991a-b). The system considered by them has a block diagonal structure and is found in general equilibrium models with nonconvex technologies or some international trade models.

Recently, somewhat different approaches for computing economic equilibria have been suggested by Luenberger and Maxfield (1995). Their algorithms relate equilibria to certain optimization problems, and these relations can be used as a basis for computing equilibria. Therefore, economically meaningful information such as benefit or surplus can be used in the process of calculation.

All the fixed point algorithms follow a path from a starting point to a zero of a map of interest. Then, in the fixed point algorithm, we can identify starting and terminating points with old and new equilibrium positions, respectively. Therefore, it may be used in comparative statics, namely, a comparison between two equilibrium positions. Especially, algorithms with the global convergence property may become powerful tools in comparative statics in the large. In fact, such approaches can be made quantitatively and qualitatively.

Economists have shown interest in what effects a policy change will have on many different markets. In this type of problem, a general equilibrium model is appropriate, but it sometimes turns out to be so complex that analytical results no longer seem possible. At this juncture, we must have recourse to simulation by computer. This 'quantitative' comparative statics was not possible until algorithms for computing an economic equilibrium were available. Shoven and Whalley, among others, made great effort to evaluate the effects of changes in taxes and tariffs (see Shoven and Whalley, 1992).

Quite a few authors have used fixed point algorithms for 'qualitative' comparative statics in the large. Fujimoto (1990) used a contraction mapping theorem, i.e., an iterative method for a fixed point, to investigate Hicksian laws in the large for the dominant diagonal case. On the other hand, Shiomura (1995-7) used a homotopy continuation method to study Hicksian laws in the large for generalized gross-substitute systems and provided a unified approach to global comparative statics. For the same reasons as stated above, Smale (1995) suggested the possibility of comparative statics in the large by making use of global Newton's method.

2 Generalizations of a basic existence theorem

Since the seminal works of Wald (1936) and von Neumann (1937) various existence theorems for economic equilibria have been obtained by many economists. Although these remarkable works have long been ignored, it is very interesting that before economists noticed the importance of a set-valued map, von Neumann had already treated its analysis, which was later developed into Kakutani's fixed point theorem.

Our purpose of this section is to give generalizations of Zangwill and Garcia

(1981b, Theorem 22.5.1). These are oriented to an economic analysis. For lower hemi-continuous maps, generalizations are made via the well-known results of Michael (1956) and Browder (1968), that is, the continuous selections theorems. On the other hand, for upper hemi-continuous maps, we obtain a generalization through the celebrated theorem of Cellina (1969), i.e., the approximate continuous selections theorem.

In the following, we use the usual Euclidean topology, unless otherwise mentioned, and consider a zero of a set-valued map: an $x \in X$ is called a zero of a set-valued map $\gamma : X \to Y$ when $0 \in \gamma(x)$. Due to slight differences in terminology, we mention the definitions about the continuity of set-valued maps following Border (1985).

Consider a set-valued map $\gamma : X \to Y$. We define the *graph* of γ by

$$Gr\gamma \equiv \{(x, y) \in X \times Y | y \in \gamma(x)\}.$$

Let $F \subset X$ and $E \subset Y$. The *image* of F under γ is defined by

$$\gamma(F) \equiv \bigcup_{x \in F} \gamma(x).$$

The *upper inverse* of E under γ is defined by

$$\gamma^{+}[E] \equiv \{x \in X | \gamma(x) \subset E\},$$

while the *lower inverse* of E under γ is defined by

$$\gamma^{-}[E] \equiv \{x \in X | \gamma(x) \cap E \neq \emptyset\}.$$

For $y \in Y$, we set

$$\gamma^{-1}(y) \equiv \{x \in X | y \in \gamma(x)\}.$$

Note that $\gamma^{-1}(y) \equiv \gamma^{-}[\{y\}]$.

Now we define the upper or lower hemi-continuity. A map $\gamma : X \to Y$ is called *upper hemi-continuous at* x if whenever x is in the upper inverse of an open set so is a neighborhood of x; and γ is *lower hemi-continuous at* x if whenever x is in the lower inverse of an open set so is a neighborhood of x. A map $\gamma : X \to Y$ is *upper hemi-continuous* (resp. *lower hemi-continuous*) if it is upper hemi-continuous (resp. lower hemi-continuous) at every $x \in X$. A map is called *continuous* if it is both upper and lower hemi-continuous.

Consider a set-valued map $\gamma : X \to Y$ with nonempty images. A single-valued map $f : X \to Y$ is called a *selection* of γ if for every $x \in X$, $f(x) \in \gamma(x)$. It should be noted that, in the above definition, the continuity of f is not required (cf. Michael, 1956).

Given $\epsilon > 0$, if there exists a single-valued map $f_\epsilon : X \to Y$ such that $Grf_\epsilon \subset B_\epsilon(Gr\gamma)$, then f_ϵ is called an *approximate selection* of γ, where $B_\epsilon(Gr\gamma)$ is an ϵ-neighborhood of $Gr\gamma$. Hereafter, we denote the boundary and the interior of X by ∂X and int X, respectively.

Now we make the assumption:

Assumption 2.1 *Let $X \subset R^n$ be a nonempty compact set with an interior point. Consider a map $\gamma : X \to R^n$. There exists an $x^0 \in \text{int } X$ such that $(x - x^0)y > 0$ for all $y \in \gamma(x)$ if $x \in \partial X$.*

Note that we assume x^0 is independent of $x \in \partial X$.

Theorem 2.1 *Let $X \subset R^n$ be a nonempty compact set with an interior point and $\gamma : X \to R^n$ have nonempty convex values and satisfy that $\gamma^{-1}(y)$ is open for each y. Then there exists a zero of γ in X if Assumption 2.1 holds.*

Proof. In view of Theorem 14.3 of Border (1985) due to Theorem 1 of Browder (1968), we can choose a continuous selection $f : X \to R^n$.

Let $h : X \times [0,1] \to R^n$ be a homotopy defined by

$$h(x,t) \equiv (1-t)(x - x^0) + tf(x),$$

where $x^0 \in \text{int } X$. Then for all $0 \leq t \leq 1$ and all $x \in \partial X$,

$$(x - x^0)h(x,t) = (1-t)||x - x^0||^2 + t(x - x^0)f(x) > 0.$$

This implies that h is boundary-free so that there exists a zero of f in X, and therefore a zero of γ. □

Theorem 2.2 *Let $X \subset R^n$ be a nonempty compact set with an interior point and $\gamma : X \to R^n$ be lower hemi-continuous with nonempty closed convex values. Then there exists a zero of γ in X if Assumption 2.1 holds.*

Proof. Applying Theorem 14.7 of Border (1985) which is due to Theorem 3.2″ of Michael (1956), we can choose a continuous selection $f : X \to R^n$. The rest of the proof is the same as stated in the previous theorem. □

The following corollary is a immediate consequence of Theorem 2.2.

Corollary 2.1 *Consider a continuous single-valued map $f : X \to R^n$, where $X \subset R^n$ is a nonempty compact set with an interior point. Suppose that for any $x \in X$, $xf(x) = 0$ and there exists an $x^0 \in \text{int } X$ such that if $x \in \partial X$, $x^0 f(x) < 0$. Then the set of zeroes of f is nonempty and compact.*

It should be noted that Corollary 2.1 does not require the convexity of X. Next we consider upper hemi-continuous maps.

Assumption 2.2 *Let $X \subset R^n$ be a nonempty compact set with an interior point. Consider a map $\gamma : X \to R^n$. For any $x \in X$, $xy = 0$ if $y \in \gamma(x)$ and there exist an $x^0 \in \text{int } X$ and an $\eta > 0$ such that $x^0 y < -\eta$ for all $y \in \gamma(x)$ if $x \in \partial X$.*

Theorem 2.3 *Let $X \subset R^n$ be a convex body and $\gamma : X \to R^n$ be upper hemi-continuous with nonempty compact convex values. Then the set of zeroes of γ is compact and nonempty if Assumption 2.2 holds.*

Proof. Using Theorem 9.2.1 of Aubin and Frankowska (1990) due to Theorem 1 of Cellina (1969), we can choose an approximate continuous selection $f^n : X \to R^n$ such that $Grf^n(x) \in B_{1/n}(Gr\gamma)$ for any integer $n > 0$.

We now introduce in $X \times R^n$ a metric ρ defined by

$$\rho((x, y), (x', y')) \equiv \max(\|x - x'\|, \|y - y'\|),$$

where $(x, y), (x', y') \in X \times R^n$. Then for any $x \in \partial X$ and $f^n(x)$, there exists a pair of $(x^n, y^n) \in Gr\gamma$ such that $\rho((x, f^n(x)), (x^n, y^n)) < 1/n$.

Define a sequence of homotopies $h^n : X \times [0,1] \to R^n$ by

$$h^n(x,t) \equiv (1-t)(x-x^0) + tf^n(x).$$

Noting that X is compact and so is $\gamma(X)$ since γ is upper hemi-continuous and compact-valued, there exist an $M > 0$ and an $N > 0$ such that $\|x - x^0\| \leq M$ for all $x \in X$ and $\|y^n\| \leq N$ for all n (see Border, 1985, Proposition 11.16). We thus have

$$\begin{aligned} |(x-x^0)f^n(x) - (x^n - x^0)y^n| &\leq \|x-x^0\| \cdot \|f^n(x) - y^n\| \\ &+ \|x - x^n\| \cdot \|y^n\| < (M+N)/n. \end{aligned}$$

Pick a point $y \in \gamma(x)$ for $x \in \partial X$ and let $V \equiv \gamma^+[\{y|x^0 y < -\eta\}]$. Since γ is upper hemi-continuous, V is an open neighborhood of x, so that there exists an n_0 such that $x_n \in V$ for all $n \geq n_0$. Thus $(x^n - x^0)y^n > \eta$ for all $n \geq n_0$ since $x^n y^n = 0$ and $x^0 y^n < -\eta$ for all $n \geq n_0$. Therefore,

$$(x-x^0)f^n(x) > \eta - (M+N)/n,$$

for all $n \geq n_0$.

Let n_1 be an integer greater than or equal to $\max\{n_0, (M+N)/\eta\}$, then we have for all $n \geq n_1$,

$$(x-x^0)h^n(x,t) = (1-t)\|x-x^0\|^2 + t(x-x^0)f^n(x) > 0,$$

if $x \in \partial X$ and $0 \leq t \leq 1$. This implies that for all $n \geq n_1$, h^n is boundary-free so that there exists an $\bar{x}_n \in X$ such that $(\bar{x}_n, 0) \in B_{1/n}(Gr\gamma)$ for all $n \geq n_1$.

Since X is compact, there exists a convergent subsequence of $\{\bar{x}_n\}$. Without loss of generality, we denote it by $\{\bar{x}_n\}$. Suppose that $\{\bar{x}_n\}$ converges to an $x^* \in X$. Then since $Gr\gamma$ is closed, $(x^*, 0) \in Gr\gamma$.

On the other hand, from the upper hemi-continuity $\gamma^{-1}(0)$ is closed in X so that it is compact. □

3 Existence of Walrasian equilibria

Let R^{n+1} denote the commodity space. For $i = 1, \ldots, m$ let $X_i \subset R^{n+1}$ denote the ith consumer's consumption set, $\omega_i \in R^{n+1}$ his private endowment. We suppose a binary relation U_i on X_i which associates to each $x_i \in X_i$ a set $U(x_i) \subset X_i$. Then $U(x_i)$ may be interpreted as the set of those object in X_i which are better than x_i. Moreover, for $j = 1, \ldots, k$ let Y_j denote the jth supplier's production set.

Put $X \equiv \sum_{i=1}^{m} X_i$, $\omega \equiv \sum_{i=1}^{m} \omega_i$ and $Y \equiv \sum_{j=1}^{k} Y_j$. Let α_j^i be the share of consumer i in the profits of supplier j. A Walrasian economy is then described by a tuple $((X_i, \omega_i, U_i), (Y_j), (\alpha_j^i))$.

An *attainable state* of the economy is a tuple $((x_i), (y_j)) \in \prod_i X_i \times \prod_j Y_j$, satisfying

$$\sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j - \omega = 0.$$

Let F denote the set of attainable states and let

$$M \equiv \{((x_i), (y_j)) \in (R^{n+1})^{m+k} | \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j - \omega = 0\}.$$

Then $F = (\prod_i X_i \times \prod_j Y_j) \cap M$. Let $\tilde{X}_i$ be the projection of F on X_i, and $\tilde{Y}_j$ be the projection of F on Y_j.

Now we make the following assumptions:

Assumption 3.1 *For each $i = 1, \ldots, m$,*

1. *X_i is closed, convex, bounded from below, and $\omega_i \in X_i$.*
2. *There exists some $\bar{x}_i \in X_i$ satisfying $\omega_i > \bar{x}_i$.*
3. *U_i has open graph.*
4. *$x_i \notin co\ U_i(x_i)$.*
5. *$x_i \in cl\ U_i(x_i)$.*

We mean by *co* $U_i(x_i)$ and *cl* $U_i(x_i)$ the convex hull and the closure of $U(x_i)$, respectively.

Assumption 3.2 *For each $j = 1, \ldots, k$,*

1. *Y_j is closed and convex and $0 \in Y_j$.*
2. *$Y \cap (-Y) = \{0\}$.*
3. *$Y \supset -R_{+}^{n+1}$.*

It should be noted that under Assumption 3.2, $Y \cap R_+^{n+1} = \{0\}$, where R_+^{n+1} denotes the nonnegative orthant of R^{n+1}. On the other hand, according to Debreu (1959, pp. 41–42) Y is closed.

Let AY be the *asymptotic cone* of Y. Then $AY \cap R_+^{n+1} = \{0\}$ since Y is a closed convex set and $0 \in Y$, and therefore, $AY \subset Y$. Keeping these remarks in mind, from Proposition 20.3 of Border (1985) we have the following lemma.

Lemma 3.1 *The set F of attainable states is compact and nonempty under Assumptions 3.1 and 3.2.*

In view of Lemma 3.1, for each consumer i there is a compact convex set K_i containing $\tilde{X}_i$ in its interior. Put $X_i^{'} \equiv K_i \cap X_i$. Likewise, for each supplier j there is a compact convex set C_j containing $\tilde{Y}_j$ in its interior. Put $Y_j^{'} \equiv C_j \cap Y_j$.

Let $S_n \equiv \{p \in R_+^{n+1} | \sum_{i=0}^{n} p_i = 1\}$, the standard n-simplex.

Assumption 3.3 *For each consumer i there is a continuous quasi-concave utility u_i satisfying $U_i(x_i) = \{x_i^{'} \in X_i^{'} | u(x_i^{'}) > u(x_i)\}$.*

Define $\gamma_j : S_n \rightarrow Y_j^{'}$ by

$$\gamma_j(p) \equiv \{y_j \in Y_j^{'} | py_j \geq py_j^{'} \quad \text{for all} \quad y_j^{'} \in Y_j^{'}\},$$

and set $\pi_j(p) \equiv max_{y_j \in Y_j^{'}} py_j$. Also define $\beta_i : S_n \rightarrow X_i^{'}$ and $\mu_i : S_n \rightarrow X_i^{'}$ by

$$\beta_i(p) \equiv \{x_i \in X_i^{'} | px_i \leq p\omega_i + \sum_j \alpha_j^i \pi_j(p)\},$$

and

$$\mu_i(p) \equiv \{x_i \in \beta_i(p) | u_i(x_i) \geq u_i(x_i^{'}) \quad \text{for all} \quad x_i^{'} \in \beta_i(p)\},$$

respectively. Then put

$$\zeta(p) \equiv \sum_i \mu_i(p) - \sum_j \gamma_j(p) - \omega.$$

The proof of the lemma below is somewhat routine. So we leave it out.

Lemma 3.2 *ζ is upper hemi-continuous with nonempty compact convex values.*

Assumption 3.4 *We assume that*

1. *$pz = 0$ for all $z \in \zeta(p)$.*
2. *There exist a $\bar{p} \in \text{int } S_n$ and an $\eta > 0$ such that $\bar{p}z > \eta$ for all $z \in \zeta(p)$ when $p \in \partial S_n$.*

Theorem 3.1 *Let the economy $((X_i, \omega_i, U_i), (Y_j), (\alpha^i_j))$ satisfy Assumptions 3.1-3.4. Then the set of zeroes of ζ, i.e., the set of Walrasian equilibria, is nonempty and compact.*

Proof. By the previous lemma, $\Gamma \equiv -\zeta$ is upper hemi-continuous with nonempty compact convex values. Applying Theorem 2.3 to Γ, we can immediately obtain the theorem. It should be noted that zero points are indeed contained in int S_n.
□

We now replace Assumptions 3.2-3.4 with the following.

Assumption 3.5 *For each $j = 1, \ldots, k$,*

1. *Y_j is closed and strictly convex and $0 \in Y_j$.*
2. *$Y \cap (-Y) = \{0\}$.*
3. *$Y \supset -R^n_+$.*

Assumption 3.6 *For each consumer i there is a continuous strictly quasi-concave utility u_i satisfying $U_i(x_i) = \{x_i' \in X_i' | u(x_i') > u(x_i)\}$.*

Assumption 3.7 *Suppose that*

1. *$pz = 0$ for all $z \in \zeta(p)$.*
2. *There exists a $\bar{p} \in \text{int } S_n$ such that $\bar{p}z > 0$ for all $z \in \zeta(p)$ when $p \in \partial S_n$.*

Then we have the following theorem.

Theorem 3.2 *Let the economy $((X_i, \omega_i, U_i), (Y_j), (\alpha^i_j))$ satisfy Assumptions 3.1 and 3.5-3.7. Then the set of zeroes of ζ, i.e., the set of Walrasian equilibria, is nonempty and compact.*

Proof. Under our assumptions, ζ is a single-valued continuous map. It follows from Corollary 2.1 that we obtain the theorem. □

4 The regular economy

Lastly we get some insights into the number of Walrasian equilibria. Now let $Z : S_n \to R^{n+1}$ be an excess demand map. Also define q and q_0 by $(p_1, \ldots, p_n)$ and $p_0 \equiv 1 - \sum_{i=1}^n p_i$, respectively. Consider a map $z : S \to R^n$ defined by

$$z_i(q) \equiv Z_i(1 - \sum_{i=1}^{n} q_i, q_1, \ldots, q_n), \qquad i = 1, \ldots, n,$$

where $S \equiv \{q \in R^n_{++} | \sum_{i=1}^n q_i < 1\}$ and R^n_{++} denotes the positive orthant of R^n. Also define f by

$$f_i(q) \equiv q_i z_i(q), \qquad i = 1, \ldots, n,$$

that is, the value of excess demand for good i.

Assumption 4.1 *Assume that*

1. $pZ(p) = 0$ *for any* $p \in \text{int } S_n$.
2. z *is of class* C^1 *on* S.
3. Dz_q *has rank* n *for all* $q \in z^{-1}(0)$, *if any.*

Assumption 4.2 *If the sequence* $\{p^\nu\} \in \text{int } S_n$ *converges to a point in* ∂S_n, *there exists a* $j \in \{0, 1, \ldots, n\}$ *such that* $\{p_j^\nu\}$ *converges to zero and* $\limsup_{\nu \to \infty} Z_j(p^\nu) = \infty$.

Assumption 4.1 implies that we consider a so-called *regular economy.* Assumption 4.2 is the desirability assumption used in Dierker (1972).

Theorem 4.1 *The economy has an odd number of equilibria if Assumptions 4.1 and 4.2 are fulfilled.*

Proof. We first construct a homotopy $h : S \times [0, 1] \to R^n$ defined by

$$h_i(q, t) \equiv (1 - t)(q_i - q_i^0) + t(-f_i(q)), \qquad i = 1, \ldots, n,$$

where $q_i^0 \equiv 1/(n + 1)$ for all $i = 1, \ldots, n$.

We show that prices on the homotopy path is never close to the boundary of S. Suppose the contrary. Then there exists a $j \in \{0, 1, \ldots, n\}$ such that $\{p_j^\nu\}$

converges to zero and $\limsup_{\nu\to\infty} Z_j(p^\nu) = \infty$. If $j \neq 0$, then for some sufficiently large ν_1

$$h_j(q^{\nu_1}, t^{\nu_1}) = (1 - t^{\nu_1})(q_j^{\nu_1} - q_j^0) + t^{\nu_1}(-f_j(q^{\nu_1})) < 0,$$

for some $t^{\nu_1} \in [0, 1]$. On the other hand if $j = 0$, define h_0 by

$$h_0(q, t) \equiv (1 - t)(q_0 - q_0^0) + t(-f_0(q)),$$

where $f_0(q) \equiv q_0 z_0(q)$ and $z_0(q) \equiv Z_0(q)$. By the same reason as stated above, $h_0(q^{\nu_2}, t^{\nu_2}) < 0$ for some sufficiently large ν_2 and for some $t^{\nu_2} \in [0, 1]$. But noting that $q_0^{\nu_2} = 1 - \sum_{i=1}^n q_i^{\nu_2}$ and $q_0^{\nu_2} z_0(q^{\nu_2}) = -\sum_{i=1}^n q_i^{\nu_2} z_i(q^{\nu_2})$, since

$$\begin{aligned} h_0(q^{\nu_2}, t^{\nu_2}) &= (1 - t^{\nu_2})(q_0^{\nu_2} - q_0^0) + t^{\nu_2}(-f_0(q^{\nu_2})) \\ &= -\sum_{i=1}^n h_i(q^{\nu_2}, t^{\nu_2}) < 0, \end{aligned}$$

there exists a $k \in \{1, \ldots, n\}$ such that $h_k(q^{\nu_2}, t^{\nu_2}) > 0$. Thus in both cases, a contradiction arises. Therefore, we may assume that prices on the homotopy path are contained in a compact subset of S, K. In addition, we may suppose that the prices are contained in int K. Therefore, h is boundary-free when we restrict h to $K \times [0, 1]$.

It follows from the homotopy invariance theorem that f has an odd number of zeroes. Since for $i = 1, \ldots, n$, $f_i(q) = 0$ if and only if $z_i(q) = 0$ as far as $q_i > 0$, the number of zeroes of f is equal to that of z. □

Instead of Assumption 4.2 we may suppose that

Assumption 4.3 *If the sequence $\{p^\nu\} \in$ int S_n converges to a point in ∂S_n, there exists a $j \in \{0, 1, \ldots, n\}$ such that $\{p_j^\nu\}$ converges to zero and $\sum_{i=0}^n Z_i(p^\nu)$ tends to infinity.*

Assumption 4.3 is due to Dierker (1974). If we suppose that Z is bounded from below on int S_n, Debreu's Assumption A implies Assumption 4.3 (see Debreu, 1970).

Theorem 4.2 *The economy has an odd number of equilibria if Assumptions 4.1 and 4.3 are fulfilled.*

Proof. Consider the homotopy defined in the proof of Theorem 4.1. Then $h_i(q,t) \leq (1-t) + t(-f_i(q))$ for $i = 1, \ldots, n$. Suppose that for some $i \in \{0, 1, \ldots, n\}$, p_i converges to zero. By Assumption 4.3, then there exist a $j \in \{0, 1, \ldots, n\}$ and a ν such that $z_j(q^\nu)$ has a sufficiently large value. We define h_0 by $h_0(q,t) \equiv -\sum_{i=1}^n h_i(q,t)$ as in the proof of Theorem 4.1.

We consider four cases, respectively. In the case that $i \neq 0$ and $j \neq 0$, when $i \neq j$, $h_j(q^\nu, t^\nu) < 0$ if $t^\nu \neq 0$, while if $t^\nu = 0$, $h_i(q^\nu, t^\nu) = q_i^\nu - q_i^0 < 0$. On the other hand, if $i = j$ then $h_i(q^\nu, t^\nu) < 0$ as stated in the proof of Theorem 4.1. In the case that $i = 0$ and $j \neq 0$, $h_j(q^\nu, t^\nu) < 0$ if $t^\nu \neq 0$, while if $t^\nu = 0$, $h_0(q^\nu, t^\nu) < 0$ so that there exists a $k \in \{1, \ldots, n\}$, such that $h_k(q^\nu, t^\nu) > 0$. In the case that $i \neq 0$ and $j = 0$, if $t^\nu \neq 0$, $h_0(q^\nu, t^\nu) < 0$ so that there exists a $k \in \{1, \ldots, n\}$ such that $h_k(q^\nu, t_1^\nu) > 0$, while if $t^\nu = 0$, $h_i(q^\nu, t^\nu) = q_i^\nu - q_i^0 < 0$. Lastly, in the case that $i = 0$ and $j = 0$, $h_0(q^\nu, t^\nu) < 0$ so that there exists a $k \in \{1, \ldots, n\}$ such that $h_k(q^\nu, t_1^\nu) > 0$.

Therefore, we may suppose that prices on the homotopy path are contained in a compact subset of S. The rest of the proof is the same as that of Theorem 4.1. □

In the above two proofs, we have shown that if Walras' law is satisfied and prices on the homotopy path are completely contained in a compact subset of S, then the regular economy has an odd number of equilibria. This is in effect Theorem 1 of Dierker (1972) (see also Dierker, 1974, Theorem 11.1).

Furthermore, we obtain the following result.

Assumption 4.4 *If the sequence* $\{p^\nu\} \in \text{int } S_n$ *converges to a point in* ∂S_n*, there exists a pair of* $(i,j), i,j \in \{0, 1, \ldots, n\}$ *such that* $\{p_j^\nu\}$ *converges to zero and* $\lim_{\nu \to \infty} Z_i(p^\nu) = \infty$*, where* i *may be equal to* j*.*

When we assume that Z is bounded from below on int S_n, Assumption 4.4 implies 4.3 and *vice versa* under Assumption 4.1.1.

Theorem 4.3 *Suppose that Assumptions 4.1.1 and 4.1.2 are satisfied. Also suppose that Assumption 4.4 holds. Then* z *is a homeomorphism of* S *onto* R^n *if* det $Dz_q \neq 0$ *at every point of* S*.*

Proof. It should be noted that if det $Dz_q \neq 0$ at every point of S, z is a local homeomorphism at every point of S.

On the other hand, under our assumptions, z is *norm-coercive* on the open convex set S, which implies that as q approaches ∂S, the Euclidean norm of $z(q)$ tends to infinity. Therefore, the theorem follows from the norm-coerciveness theorem (see Ortega and Rheinboldt, 1970, pp. 136–137). □

The above theorem, in particular, shows that a regular economy has uniquely an equilibrium price vector provided that det Dz_q has a nonzero constant sign at all $q \in S$ with the desirability assumption. It is, however, sufficient for the uniqueness that det Dz_q has a nonzero constant sign at all Walrasian equilibria together with the desirability assumption (see Nishimura, 1978, Theorem 1).

References

Aubin, J. -P. and Frankowska, H. (1990). *Set-Valued Analysis*, Birkhäuser, Boston, MA.

Border, K. C. (1985). *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge.

Browder, F. E. (1968). The fixed point theory of multi-valued mappings in topological vector spaces, *Mathematische Annalen*, **177**, 283–301.

Cellina, A. (1969). Approximation of set valued functions and fixed point theorems, *Annali di Mathematica Pura ed Applicata*, **82**, 17–24.

Debreu, G. (1959). *Theory of Value*, Yale University Press, New Haven.

Debreu, G. (1970). Economies with a finite set of equilibria, *Econometrica*, **38**, 387–392.

Dierker, E. (1972). Two remarks on the number of equilibria of an economy, *Econometrica*, **40**, 951–953.

Dierker, E. (1974). *Topological Methods in Walrasian Economics*, Springer-Verlag, Berlin.

Doup, T., van der Laan, G. and Talman, A. J. J. (1987). $(2^{n+1} - 2)$-ray algorithm: a new simplicial algorithm to compute economic equilibria, *Mathematical Programming*, **39**, 241–252.

Eaves, B. C. (1976). A finite algorithm for the linear exchange model, *Journal of Mathematical Economics*, **3**, 197–203.

Fujimoto, T. (1988). The existence, uniqueness and global stability of competitive equilibrium under gross substitutability: a difference equation approach, *Economic Studies Quarterly*, **39**, 271–276.

Fujimoto, T. (1990). Hicksian laws of comparative statics for the dominant diagonal case, *Economic Studies Quarterly*, **41**, 282–284.

Gabay, D. and Moulin, H. (1980). On the uniqueness and stability of Nash-equilibria in noncooperative games, 271–298, in: A. Bensoussan, P. Kleindorfer, and C. S. Tapiero (eds), 1980, *Applied Stochastic Control in Econometrics and Management Science*, North-Holland, Amsterdam.

Gale, D. (1964). On equilibrium for a multi-sector model of income propagation, *International Economic Review*, **5**, 185–200.

Hadar, J. (1965). A note on dominant diagonals in stability analysis, *Econometrica*, **33**, 442–444.

Hadar, J. (1966). Stability of oligopoly with product differentiation, *Review of Economic Studies*, **33**, 57–60.

Kamiya, K. (1990). A globally stable price adjustment process, *Econometrica*, **58**, 1481–1485.

Kamiya, K. (1991a). Computation of equilibria in an economy with increasing returns to scale technologies, *Mathematical Programming*, **49**, 253–261.

Kamiya, K. (1991b). Efficient algorithms for solving systems of nonlinear equations with a block diagonal structure, *Mathematics of Operations Research*, **16**, 566–579.

Kellogg, R. B., Li, T. Y. and Yorke, J. (1976). A constructive proof of the Brower fixed-point theorem and computational resutls, *SIAM Journal of Numerical Analysis*, **13**, 473–483.

van der Laan, G. (1985). The computation of general equilibrium in economies with a block diagonal pattern, *Econometrica*, **53**, 659–665.

Luenberger, D. G. and Maxfield, R. R. (1995). Computing economic equilibria using benefit and surplus functions, *Computational Economics*, **8**, 47–64.

Mansur, A. and Whalley, J. (1982). A decomposition algorithm for general equilibrium computation with application to international trade models, *Econometrica*, **50**, 1547–1557.

Mathiesen, L. (1985). Computation of economic equilibria by a sequence of linear complementarity problems, *Mathematical Programming Study*, **23**, 144–162.

Mayerson, R. B. (1981). An algorithm for computing equilibria in a linear monetary economy, *Mathematical Programming*, **21**, 182–189.

Michael, E. (1956). Continuous selections, I, *Annals of Mathematics*, **63**, 361–382.

von Neumann, J. (1937). Über ein ökonomisches gleichungssystem und eine verallgemeinerung des Browerschen fixpunktsatzes, *Ergebnisse eines Mathematischen Kolloquiums*, **8**, 73–83, translated as 'A model of general economic equilibrium' in 1945-46, *Review of Economic Studies*, **13**, 1–9.

Nishimura, K. (1978). A further remark on the number of equilibria of an economy, *International Economic Review*, **19**, 679–685.

Okuguchi, K. (1970). On the stability of the Cournot oligopoly equilibrium, *Economic Studies Quarterly*, **21**, 56–59.

Okuguchi, K. (1983). The Cournot oligopoly and competitive equilibria as solutions to non-linear complementarity problems, *Economic Letters*, **12**, 127–133.

Okuguchi, K. and Szidarovszky, F. (1990). *The Theory of Oligopoly with Multi-Product Firms*, Springer-Verlag, Berlin.

Ortega, J. M. and Rheinboldt, W. C. (1970). *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York.

Preckel, P. V. (1985). Alternative algorithms for computing economic equilibria, *Mathematical Programming Study*, **23**, 163–172.

Scarf, H. (1967). The approximation of fixed points of a continuous mapping, *SIAM Journal of Applied Mathematics*, **15**, 1328–1343.

Scarf, H. (1973). *The Computation of Economic Equilibria*, Yale University Press, New Haven.

Shiomura, T. (1995). Comparative statics in the large based on a fixed point algorithm, *Journal of Economics*, **62**, 77–92.

Shiomura, T. (1996). *The Path-Following Approach to Economic Equilibria*, Unpublished Doctoral Thesis, Kobe University.

Shiomura, T. (1997). On the Hicksian laws of comparative statics in the large for the Hicksian case: the path-following approach using an alternative homotopy, *Computational Economics*, to appear.

Shoven, J. B. and Whalley, J. (1992). *Applying General Equilibrium*, Cambridge University Press, Cambridge.

Smale, S. (1976). A convergent process of price adjustment and global Newton methods, *Journal of Mathematical Economics*, **3**, 107–120.

Smale, S. (1995). Comparative statics and algorithms for finding economic equilibria, 267–270 in: T. Maruyama and W. Takahashi (eds), 1995, *Nonlinear and Convex Analysis in Economic Theory*, Springer-Verlag, Berlin.

Thorlund-Petersen, L. (1985). Fixed point iterations and global stability in ecomomics, *Mathematics of Operations Research*, **10**, 642–649.

Todd, N. J. (1976). *The Computation of Fixed Points and Applications*, Springer-Verlag, Berlin.

Uzawa, H. (1960). Walras' tâtonnement in the theory of exchange, *Review of Economic Studies*, **27**, 182–194.

Wald, A. (1936). Über einige gleichungssyteme der mathematischen ökonomie, *Zeitschrift für Nationalöknomie*, **7**, 637–670, translated as 'On some systems of equations of mathematical economics' in 1951, *Econometrica*, **19**, 368–403.

Zangwill, W. I. and Garcia, C. B. (1981a). Equilibrium programming: the path-following approach and dynamics, *Mathematical Programming*, **21**, 262–289.

Zangwill, W. I. and Garcia, C. B. (1981b). *Pathways to Solutions, Fixed Points, and Equilibria*, Prentice-Hall, Englewood Cliffs, NJ.