A Homotopy Method for a Comparison between a System and its Sub-system

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Abstract

The paper discusses, using the path-following algorithm, a comparison between two equilibrium positions of a system and its sub-system. We consider from a global viewpoint an extension of the strong Le Chatelier-Samuelson principle to an economy containing gross-complements. We also briefly discuss a time-honored problem, Cournot's conjecture. The paper suggests that the path-following approach is useful for comparative statics in the large when not only simple parametric changes but also more complicated ones in a system have occurred.

1 Introduction

Economists have great interest in the changes of an equilibrium position when a set of policy parameters or consumer taste has altered. As was shown by Shiomura (1995, 1997), the path-following algorithm discussed by Garcia and Zangwill (1979), which is a fixed-point algorithm using a homotopy, makes it possible to study this problem from a global view point in an easy and systematic manner.

In some cases, we are concerned with such a problem as a comparison between an equilibrium of a system and that of the sub-system. That is, a comparison between two solutions to systems of equations

$$f_i(x_1, \ldots, x_n; \alpha) = 0, \qquad i = 1, \ldots, n, \tag{1}$$

$$f_i(x_1, \ldots, x_m, \bar{x}_{m+1}, \ldots, \bar{x}_n; \alpha) = 0, \qquad i = 1, \ldots, m, \tag{2}$$

where $1 \leq m < n$; besides, $\alpha \in R^l$ and $\bar{x}_j, j = m+1, \ldots, n$ are given exogenously. The strong Le Chatelier-Samuelson principle discussed by Samuelson (1947) is a typical example of this type. A classical problem traced back to Cournot (1838), the quasi-competitiveness in an oligopoly market, is also included in the above problem.

The present paper investigates the global strong Le Chatelier-Samuelson principle making use of the path-following approach, and extends it to an economy containing complementary commodities, the Morishima case. Subsequently, we suggest a general procedure for a comparison between a system and its sub-system, taking Cournot's conjecture as an illustration.

2 The strong Le Chatelier-Samuleson principle

The Le Chatelier-Samuelson principle was originally concerned with a problem of thermochemical equilibrium and was introduced into economic theory by Samuelson (1947). The principle was argued in connection with extremum problems. Later, Samuelson (1960) recast it on general systems which are not directly governed by extremization. Although the principle was stated in a somewhat ambiguous setting, Eichhorn and Oettli (1972) refined it in terms of weak and strong versions of the principle. In the present paper, we confine our attention to the latter case only.

The local and global versions have also appeared in the literature. The former was discussed extensively by Kusumoto (1976), the later by Morishima (1964), Sandberg (1974) and Fujimoto (1980).

Now we are concerned with an extension of the global strong principle to an economy containing gross-complements, so we reformulate it for that purpose. Let $n \geq 2$ and put $I \equiv \{1, \ldots, n\}$. Furthermore, let U and T denote the given nonempty proper subsets of I such that $U \subset T$ and $U \neq T$. Suppose that the system of equations (1) has solutions x^0 and x^1 according as α equals to α^0 or α^1. Also suppose that when $\alpha = \alpha^1$, the sub-system (2), in which $n - m$ is equivalent to the number of the elements of U (resp. T), has a solution x^u (resp. x^t) under the constraints that $\bar{x}_j = x_j^0$ for all $j \in U$ (resp. $j \in T$) and for at least one $j \in U$ (resp. $j \in T$) $\bar{x}_j \neq x_j^1$. Then, the global strong Le Chatelier-Samuelson principle states that

$$|x_i^1 - x_i^0| \geq |x_i^u - x_i^0| \geq |x_i^t - x_i^0|, \qquad i \in I - T,$$

where sgn $(x_i^1 - x_i^0) =$ sgn $(x_i^u - x_i^0) =$ sgn $(x_i^t - x_i^0)$ for all $i \in I - T$ (cf. Morishima (1964) and Fujimoto (1980)).

In the following, we assume that there exist $n + 1$ commodities, labelled $0, 1, \ldots, n$, and commodity 0 is chosen as the numeraire. Let $e_i(p; \alpha)$ denote the excess demand function for commodity i, where $p \equiv (p_1, \ldots, p_n)$ stands for a normalized price vector and $\alpha \in R$ a shift parameter.

Assumption 2.1 *We make the following assumptions.*

1. *The Walras law is satisfied, i.e.,* $\sum_{i=0}^{n} p_i e_i(p; \alpha) \equiv 0$*, where* $p_0 \equiv 1$.
2. *Each* $e_i(p; \alpha)$ *is assumed to be continuously differentiable for any* $p > 0$.
3. *If* p_i *tends to zero,* $e_i(p; \alpha^k) > 0$ $(k = 0, 1)$*, while if* p_i *tends to infinity,* $e_i(p; \alpha^k) < 0$ $(k = 0, 1)$.
4. *The parameter shifts from* α^0 *to* α^1*, such that* $e_l(p^0; \alpha^1) > 0, l \neq 0$*, and, for any* $p > 0$*,* $e_i(p; \alpha^1) = e_i(p; \alpha^0), i \neq 0, l$*, where* p^k *is a solution to* $e(p; \alpha^k) = 0$.
5. *There exists an equilibrium price* p^0 *such that* $e(p; \alpha^0) = 0$.
6. $\sum_{j=0}^{n} e_{ij} p_j = 0$ *for any* $p > 0$*, where* $e_{ij} \equiv \partial e_i / \partial p_j$.

Let M be a nonempty proper subset of I and re-label the indices of commodities such that $M \equiv \{1, \ldots, m\}$ and $I - M \equiv \{m+1, \ldots, n\}$. In addition, we denote by x and y vectors consisting of the first m and remaining $n - m$ elements of p, respectively. Define $f_i(x, y; \alpha)$ as $-e_i(p; \alpha)$ for all $i \in M$ and put $y_j^0 \equiv p_j^0$ and $y_j^1 \equiv p_j^1$ for all $j \in I - M$. We consider three zero points of maps

$$
\begin{aligned}
f^0(x) &\equiv f(x, y^0; \alpha^0), && (3)\\
f^1(x) &\equiv f(x, y^1; \alpha^1), && (4)\\
\bar{f}(x) &\equiv f(x, \bar{y}; \alpha^1), && (5)
\end{aligned}
$$

where $\bar{y}$ is given exogenously. It should be noted that, by our definition, the first m elements of p^0 and p^1 become zeroes of (3) and (4), denoted by x^0 and x^1, respectively. We also denote a zero of (5) by $\bar{x}$.

Consider two homotopies,

$$
\begin{aligned}
h^1(x, t) &\equiv (1 - t) f^0(x) + t\bar{f}(x),\\
h^2(x, t) &\equiv (1 - t)\bar{f}(x) + t f^1(x),
\end{aligned}
$$

defined on $Q \equiv X \times [0, 1]$, where X is a hyperrectangle of R^m_{++}, the positive orthant of R^m. For convenience sake, we call h^k 'regular' if $D_x h^k$ has full rank for all $(x, t) \in Q$, where $D_x h^k$ is the Jacobian matrix of h^k with respect to $x \in X$. Further, h^k is called 'boundary-free' at t if $x \notin \partial X$ for any x such that $h^k(x, t) = 0$, where ∂X is the boundary of X (see Zangwill and Garcia (1981)).

A theorem on the path-following algorithm tells us that there exists a continuously differentiable 'homotopy-path' which starts from a solution to $h^k(x, 0) = 0$ and terminates at a solution to $h^k(x, 1) = 0$ if h^k is regular and boundary-free at all $0 \leq t \leq 1$. Then, differentiating $h^k(x, t) = 0, k = 1, 2$ with respect to the arc length of the path we obtain two differential equations

$$
\begin{aligned}
\dot{x} &= -\dot{t} D_x {h^1}^{-1} \cdot (\bar{f} - f^0), && (6)\\
\dot{x} &= -\dot{t} D_x {h^2}^{-1} \cdot (f^1 - \bar{f}), && (7)
\end{aligned}
$$

where a dot stands for a differentiation with respect to the arc length. A similar argument to that of Shiomura (1995) makes sure that if $D_x h^k$ is nonsingular, paths connecting x^0 to $\bar{x}$ and $\bar{x}$ to x^1 can be constructed, and $\dot{t} > 0$ along them under Assumption 2.1.

We call $J \equiv [e_{ij}]$ Metzlerian if all its diagonal entries are negative and all its off-diagonal entries are nonnegative.

Assumption 2.2 *For all $i = 1, \ldots, n$, $e_{i0} > 0$ for any p and any α.*

Lemma 2.1 *Suppose that J is Metzlerian for any p and any α. Also suppose that Assumptions 2.1 and 2.2 are satisfied. Then, $D_x h^k$ is nonnegatively invertible and its inverse has all positive diagonal entries if $0 \leq t \leq 1$.*

Proof. Put $Df^0 \equiv [f^0_{ij}]$ and $D\bar{f} \equiv [\bar{f}_{ij}]$. By Assumptions 2.1.6 and 2.2, for all $i \in M$

$$\begin{aligned}\sum_{j \in M} \{(1-t)f^0_{ij} + t\bar{f}_{ij}\}x_j = & - (1-t)(f^0_{i0} + \sum_{j \in I-M} f^0_{ij} y^0_j) \\ & - t(\bar{f}_{i0} + \sum_{j \in I-M} \bar{f}_{ij}\bar{y}_j) > 0,\end{aligned}$$

if $0 \leq t \leq 1$. It follows from Hawkins-Simon's theorem that $D_x h^1$ is nonnegatively invertible and the diagonal elements of $D_x h^{1^{-1}}$ are all positive. Similarly for $D_x h^2$. □

We strengthen a part of Assumption 2.1.4 as follows.

Assumption 2.3 *For any $p > 0$, $e_l(p, \alpha^0) < e_l(p, \alpha^1)$ for an $l \in M$.*

Theorem 2.1 *Suppose that J is Metzlerian for any p and any α. Then, under Assumptions 2.1 to 2.3, $0 < x^0 \leq \bar{x} \leq x^1$ and $x^0_l < \bar{x}_l$ if $y^0 \leq \bar{y} \leq y^1$ and $\bar{y} \neq y^1$.*

Proof. By Shiomura (1995, Theorem 4.2), we first note that there exists uniquely an equilibrum price vector $p^1 > 0$ such that $p^1 \geq p^0$ and $p^1_l > p^0_l$ when $\alpha = \alpha^1$ under Assumptions 2.1 and 2.2. The uniqueness and positivity of p^0 are also assured.

Since $\bar{y} \geq y^0$ and $f_{ij} \leq 0$ for all $i \neq j$, we obtain for any x

$$f_i(x, \bar{y}; \alpha^0) \leq f_i(x, y^0; \alpha^0) = f^0_i(x), \qquad i \in M,$$

while, in view of Assumptions 2.1.4 and 2.3, we have for any x

$$\bar{f}_i(x) = f_i(x, \bar{y}; \alpha^1) \leq f_i(x, \bar{y}; \alpha^0), \qquad i \in M,$$

with a strict inequality for only l. It follows from (6) and Lemma 2.1 that $0 < x^0 \leq \bar{x}$ and $x_l^0 < \bar{x}_l$.

On the other hand, we have for any x

$$\bar{f}_i(x) = f_i(x, \bar{y}; \alpha^1) \geq f_i(x, y^1; \alpha^1) = f_i^1(x), \qquad i \in M,$$

since $\bar{y} \leq y^1$. Then, by Lemma 2.1 together with (7), $\bar{x} \leq x^1$. □

The theorem can be applied recursively to systems in which more of the endogenous variables are fixed, and therefore, Theorem 2.1 turns out to be a differentiable variant of Fujimoto's (1980) Theorem 3. It is noteworthy, however, that the existence of solutions to the sub-systems is shown in the proof owing to an argument about a homotopy continuation method. It should be noticed also that the result in the theorem holds good with strict inequalities if we suppose that J is Metzlerian and indecomposable for any p and α, and that for any α there exists a $k \in M$ such that $e_k(p;\alpha) < e_k(p';\alpha)$ if $p_i = p'_i, i \in M$ and $p_j \leq p'_j, j \in I - M$ with a strict inequality holding for at least one $j \in I - M$.

Now we attempt to extend Theorem 2.1 to the Morishima case. Suppose that the nonnumeraire goods are divided, after suitable re-labelling of goods, into nonoverlapping groups, $K \equiv \{1, \ldots, k\}$ and $L \equiv \{k+1, \ldots, n\}$ $(n \geq 3)$, such that any two goods belonging to the same group are substitutes for each other and any two goods belonging to different groups are complementary with each other. In other words, we suppose that J is a Morishima matrix such that

$$\begin{aligned} e_{ij} &\geq 0, \quad i \neq j;\ i,j \in K \text{ or } i,j \in L, \\ e_{ij} &\leq 0, \quad i \in K, j \in L \text{ or } i \in L, j \in K, \\ e_{ii} &< 0, \quad i = 1, \ldots, n. \end{aligned}$$

We also suppose Morishima's (N') which ensures the global stability of the Morishima case (see Morishima (1970)).

Assumption 2.4 *For any p and any α,*

$$\begin{aligned} e_{i0} + 2\sum_{j \in L} e_{ij} p_j &> 0, \qquad i \in K, \\ e_{i0} + 2\sum_{j \in K} e_{ij} p_j &> 0, \qquad i \in L. \end{aligned}$$

Note that Assumption 2.4 implies 2.2. Let R and S be two nonempty subsets such that $R \equiv \{1, \ldots, r\} \subset K$ and $S \equiv \{k+1, \ldots, s\} \subset L$, and that $R \cup S$ becomes a proper subset of I.

Lemma 2.2 *Suppose that Assumptions 2.1 and 2.4 are fulfilled. Then, if J is a Morishima matrix with all nonzero entries for any p and any α, and that the sign patterns of its elements remain unchanged irrespective of the values of p and α, $D_x h^k$ is invertible and its inverse has the form of*

$$\begin{pmatrix} H_{RR} & -H_{RS} \\ -H_{SR} & H_{SS} \end{pmatrix},$$

where each sub-matrix $H_{ij}, i, j \in \{R, S\}$ has all positive entries.

Proof. Put $J^k \equiv -D_x h^k, k = 1, 2$. Then, J^k is a Morishima matrix with all nonzero entries if $0 \leq t \leq 1$. Define the matrix

$$P \equiv \begin{pmatrix} I_R & 0 \\ 0 & -I_S \end{pmatrix},$$

where I_R and I_S are the identity matrices of order r and $s - k$, respectively. Then PJ^kP^{-1} is Metzlerian if $0 \leq t \leq 1$. Noting that for any p and any α

$$\begin{aligned}
\sum_{j \in R} e_{ij}p_j + \sum_{j \in S}(-e_{ij}p_j) &= -(\sum_{j \in K-R} e_{ij}p_j + e_{i0} + 2\sum_{j \in S} e_{ij}p_j + \sum_{j \in L-S} e_{ij}p_j) \\
&< -(\sum_{j \in K-R} e_{ij}p_j + e_{i0} + 2\sum_{j \in L} e_{ij}p_j) < 0, \quad i \in R, \\
\sum_{j \in R}(-e_{ij}p_j) + \sum_{j \in S} e_{ij}p_j &= -(\sum_{j \in L-S} e_{ij}p_j + e_{i0} + 2\sum_{j \in R} e_{ij}p_j + \sum_{j \in K-R} e_{ij}p_j) \\
&< -(\sum_{j \in L-S} e_{ij}p_j + e_{i0} + 2\sum_{j \in K} e_{ij}p_j) < 0, \quad i \in S,
\end{aligned}$$

under Assumptions 2.1.6 and 2.4, by applying Hawkins-Simon's theorem to $-PJ^kP^{-1}$ we obtain the lemma if $0 \leq t \leq 1$. □

Theorem 2.2 *Suppse that J is a Morishima matrix with all nonzero entries for any p and any α, and that the sign patterns of its elements remain unchanged irrespective of the values of p and α. When $l \in R$ (resp. $l \in S$), if $y^0 = \bar{y}$ and if*

$\bar{y}_j < y_j^1, j \in K - R$ (resp. $j \in L - S$) and $\bar{y}_j > y_j^1, j \in L - S$ (resp. $j \in K - R$), then $0 < x_i^0 < \bar{x}_i < x_i^1$ for all $i \in R$ (resp. $i \in S$) and $x_i^0 > \bar{x}_i > x_i^1 > 0$ for all $i \in S$ (resp. $i \in R$) under Assumptions 2.1, 2.3 and 2.4.

Proof. It follows from Shiomura (1995, Theorem 4.1) that there exists uniquely an equilibrium price vector $p^1 > 0$ such that $p_l^1 > p_l^0$ and $p_j^1 > p_j^0$ if good j is a substitute of good l, while $p_j^1 < p_j^0$ if good j is a complement of good l. The uniqueness and positivity of p^0 are also verified.

Since $\bar{y} = y^0$, we obtain for any x

$$f(x, \bar{y}; \alpha^0) = f(x, y^0; \alpha^0) = f^0(x),$$

while, in view of Assumptions 2.1.4 and 2.3, we have for any x

$$\bar{f}(x) = f(x, \bar{y}; \alpha^1) \leq f(x, \bar{y}; \alpha^0),$$

where a strict inequality holds for only l. We thus have for any x

$$\bar{f}(x) \leq f^0(x),$$

with a strict inequality for only l. It follows from (6) and Lemma 2.2 that if $l \in R$

$$\begin{aligned} 0 < x_i^0 < \bar{x}_i, &\qquad i \in R, \\ x_i^0 > \bar{x}_i > 0, &\qquad i \in S, \end{aligned}$$

while if $l \in S$,

$$\begin{aligned} x_i^0 > \bar{x}_i > 0, &\qquad i \in R, \\ 0 < x_i^0 < \bar{x}_i, &\qquad i \in S. \end{aligned}$$

On the other hand, if $\bar{y}_j < y_j^1, j \in K - R$ and $\bar{y}_j > y_j^1, j \in L - S$, we have for any x

$$\begin{aligned} \bar{f}(x) > f^1(x), &\qquad i \in R, \\ \bar{f}(x) < f^1(x), &\qquad i \in S, \end{aligned}$$

while if $\bar{y}_j > y_j^1, j \in K - R$ and $\bar{y}_j < y_j^1, j \in L - S$, the inequalities are all reversed. It follows from (7) and Lemma 2.2 that

$$\begin{aligned} \bar{x}_i < x_i^1, &\qquad i \in R, \\ \bar{x}_i > x_i^1, &\qquad i \in S, \end{aligned}$$

if $\bar{y}_j < y_j^1, j \in K - R$ and $\bar{y}_j > y_j^1, j \in L - S$, whereas

$$\begin{aligned} \bar{x}_i &> x_i^1, & i \in R, \\ \bar{x}_i &< x_i^1, & i \in S, \end{aligned}$$

if $\bar{y}_j > y_j^1, j \in K - R$ and $\bar{y}_j < y_j^1, j \in L - S$. Consequently, we have the desired result. □

Theorem 2.2 holds good recursively, so that it becomes a global extension of the strong Le Chatelier-Samuelson principle for a gross-substitute economy to an economy containing complementary commodities.

3 Entry in an oligoply market

The previous method is somewhat specific to the problems considerd, so we next suggest a more general procedure for a comparison between a system and its sub-system. Again, consider the systems of equations (1) and (2). Let $x \equiv (x_1, \ldots, x_m)$, $y \equiv (x_{m+1}, \ldots, x_n)$ and $\bar{y} \equiv (\bar{x}_{m+1}, \ldots, \bar{x}_n)$. Suppose that (1) has uniquely a solution (x^0, y^0) and (2) a solution x^1 when $y = \bar{y}$. Also suppose that we are concerned with a comparison between (x^0, y^0) and $(x^1, \bar{y})$.

For that purpose, we introduce $n - m$ maps $\psi_j, j = m + 1, \ldots, n$ such that $\psi(y) = 0$ if and only if $y = \bar{y}$. Then make a homotopy

$$h(x, y, t) = (1 - t)f^0(x, y) + tf^1(x, y),$$

where $f^0 \equiv (f_1, \ldots, f_n)$ and $f^1 \equiv (f_1, \ldots, f_m, \psi_{m+1}, \ldots, \psi_n)$. Thus, if we can construct a homotopy-path starting from $(x^0, y^0, 0)$ and terminating at $(x^1, \bar{y}, 1)$, we can do the comparison by observing the gradient of the path.

As an illustration of that use, we consider a time-honored problem, Cournot's conjecture. Namely, an increase in the number of oligopolists increases the total output, and therefore decreases the price when all of the oligopolists are confronted with the demand function with negative slope (see Cournot (1838)). The local justification was made by, e.g., Okuguchi (1973), while Szidarovszky and Yakowitz (1982) showed that the conjecture holds good globally under fairly weak assumptions.

We imagine an oligopolistic market in which there exist N $(N \geq 2)$ firms producing homogeneous goods. Let $p(\sum x_i)$ be an inverse demand function of the market, where x_i is the output of the ith firm. We assume that x_i can vary in a bounded closed interval $\Omega_i \equiv [0, \omega_i]$. Put $x \equiv (x_1, \ldots, x_N)$ and $\Omega^N \equiv \prod_i^N \Omega_i$. We denote the cost function of the ith firm by $C_i(x_i)$.

Let X be an arbitrary subset of R^N. Hereafter, a map f is called continuously differentiable on X if there exist an open set U containing X and continuously differentiable map F that coincides with f throughout $U \cap X$. Other cases are defined similarly.

Assumption 3.1 *We now reproduce the assumptions made by Okuguchi (1973).*

1. *p is twice continuously differentiable and $p' < 0$ for any $x \in \Omega^N$.*
2. *For all i, C_i is twice continuously differentiable and satisfies the condition that $C_i(0) = 0$.*
3. *For all i, $C_i'' > p'$ for any $x \in \Omega^N$.*
4. *For all i, $p' + x_i p'' < 0$ for any $x \in \Omega^N$.*
5. *For all i, $C_i'(0) < x_i p' + p < C_i'(\omega_i)$ for any $x \in \Omega^N$.*

Under Assumption 3.1, we can show that there exists uniquely a Cournot equilibrium in the interior of Ω^N (see Appendix, Theorem A.1). At the equilibrium, the following equation holds.

$$p(\sum x_i) + x_i p'(\sum x_i) - C_i'(x_i) = 0, \qquad i = 1, \ldots, N.$$

Let x^0 and x^1 be Cournot equilibria when $N = n+1$ and $N = n$, respectively, and denote the functions $C_i' - (p + x_i p')$ by $f_i, i = 1, \ldots, n+1$. Put $f^0 \equiv (f_1, \ldots, f_n, f_{n+1})$ and $f^1 \equiv (f_1, \ldots, f_n, \psi_{n+1})$. Then, construct a homotopy

$$\begin{aligned} h(x,t) &\equiv (1-t) f^0(x) + t f^1(x), \\ &= (f_1, \ldots, f_n, (1-t) f_{n+1} + t\psi_{n+1}) \end{aligned}$$

defined on $\Omega^{n+1} \times [0,1]$. We set $\psi_{n+1}(x_{n+1}) \equiv C_{n+1}'(x_{n+1}) - C_{n+1}'(0)$. It should be noted that when $C_{n+1}''(x_{n+1}) > 0$ for any $x_{n+1} \in \Omega_{n+1}$, $h(x,1) = 0$ if and only if $x = (x^1, 0)$.

Theorem 3.1 *Suppose that $C''_i(x_i) > 0$ for any $x_i \in \Omega_i$. Then, under Assumption 3.1 other than 3.1.3, $0 < x_i^0 < x_i^1$ for $i = 1, \ldots, n$ and $\sum_i^{n+1} x_i^0 > \sum_i^n x_i^1$.*

Proof. We first note that if $C_i'' > 0$ with Assumption 3.1.1, then Assumption 3.1.3 holds. From the definition of the homotopy, $D_x h$ is a nonnegative square matrix for any $x \in \Omega^{n+1}$ and any $t \in [0, 1]$, and indeed a positive square matrix for any $x \in \Omega^{n+1}$ and any $t \in [0, 1)$.

We denote $D_x h$ by $[h_{ij}]$, and let J be any nonempty proper subset of I. Given k ($k \notin J$), for any $x \in \Omega^{n+1}$ and any $t \in [0, 1]$ we have the inequalities

$$\frac{1}{\sharp J} \sum_{j \in J} h_{ij} > h_{ik}, \qquad i \in J,$$
$$\frac{1}{\sharp J} \sum_{j \in J} h_{ij} \leq h_{ik}, \qquad i \in \bar{J},$$

with a strict inequality for $i = k$ since $h_{ii} > h_{ij} = h_{ik}$ for all distinct i, j, k, where $\sharp J$ is the number of the elements of J. It follows from Lemma A.2 in the Appendix that $D_x h$ is nonsingular for any $x \in \Omega^{n+1}$ and any $t \in [0, 1]$. On the other hand, in view of Assumption 3.1.5, the boundary-free condition holds for $0 \leq t < 1$.

Consider the sequence $\tau^k \to 0$, where all $\tau^k > 0$ and define the sequence of homotopies such that

$$H^k(x, t) \equiv h(x, t/(1 + \tau^k)).$$

Then H^k is regular and boundary-free at all $0 \leq t \leq 1$. Therefore, there exists a homotopy-path starting from $(x^0, 0)$ and terminating at $(x^{1k}, 1)$, where x^{1k} is a solution to $H^k(x, 1) = 0$.

Differentiating the path with respect to the arc length, we get

$$\dot{x}_i = -\dot{t}(\psi_{n+1} - f_{n+1}) H^*_{in+1}/(1 + \tau^k),$$

for $i = 1, \ldots, n + 1$, where H^*_{ij} is the (i, j)th element of $D_x H^{k^{-1}}$. As noted before, we can presuppose that $\dot{t} > 0$ on the homotopy-path. Note also that $\psi_{n+1} - f_{n+1} > 0$ under Assumption 3.1.5.

Using Lemma A.2 again, we can verify that $H^*_{in+1} < 0$ for all $i \neq n + 1$. Moreover, we can show that $\sum_i^{n+1} \dot{x}_i < 0$. To see this, we consider the sign of

$\sum_i^{n+1} H^*_{in+1}$ along the homotopy-path. Let

$$\begin{aligned} A_i &\equiv C_i'' - p', \\ a_i &\equiv -(p' + x_i p''), \end{aligned}$$

for $i = 1, \ldots, n$, and denote the (i,j)th element of $D_x H^k$ by H^k_{ij}. Then, for $i = 1, \ldots, n$, $H^k_{ii} = A_i + a_i$ and $H^k_{ij} = a_i$ for all $j \neq i$. If $i \neq n+1$, $\sum_l H^k_{il} H^*_{ln+1} = a_i \sum_l H^*_{ln+1} + A_i H^*_{in+1} = 0$. We thus have $\sum_l H^*_{ln+1} > 0$, since $H^*_{in+1} < 0$, $A_i > 0$, and $a_i > 0$ for all i. Therefore, $x_i^0 < x_i^{1k}$ for all $i = 1, \ldots, n$ and $\sum_i^{n+1} x_i^0 > \sum_i^{n+1} x_i^{1k}$.

Let the sequence x^{1k} have a cluster point x^*. Then, taking subsequences if necessary,

$$\lim_{k \to \infty} H^k(x^{1k}, 1) = \lim_{k \to \infty} h(x^{1k}, 1/(1 + r^k)) = h(x^*, 1) = 0,$$

so that $x^* = (x^1, 0)$, and therefore, $x_i^0 \leq x_i^1$ for all $i = 1, \ldots, n$ and $\sum_i^{n+1} x_i^0 \geq \sum_i^n x_i^1$.

We finally show that the inequalities above, in fact, hold good strictly. Let $s^0 \equiv \sum_i^{n+1} x_i^0$ and $s^1 \equiv \sum_i^n x_i^1$. If $s^0 = s^1$, then there is an $i \neq n+1$ such that $x_i^0 < x_i^1$ because $x^0_{n+1} > 0$. This implies that

$$\begin{aligned} 0 &= p(s^1) + x_i^1 p'(s^1) - C_i'(x_i^1) < p(s^1) + x_i^0 p'(s^1) - C_i'(x_i^0) \\ &= p(s^0) + x_i^0 p'(s^0) - C_i'(x_i^0) = 0, \end{aligned}$$

since $p' < 0$ and $C_i'' > 0$. This is a contradiction. If $x_i^0 = x_i^1$ for some $i \neq n+1$,

$$\begin{aligned} 0 &= p(s^1) + x_i^1 p'(s^1) - C_i'(x_i^1) = p(s^1) + x_i^0 p'(s^1) - C_i'(x_i^0) \\ &> p(s^0) + x_i^0 p'(s^0) - C_i'(x_i^0) = 0, \end{aligned}$$

since $s^0 > s^1$ and $p' + x_i p'' < 0$. Again, we obtain a contradiction. The proof is thus complete. □

Theorem 3.1 is a global extension of Okuguchi (1973), but a special case of Szidarovszky and Yakowitz (1982, Theorem 4).

4 Concluding remarks

We thus far have studied comparative statics in the large based on a fixed-point algorithm. In Shiomura (1995, 1997), we investigated from a global view point the Hicksian laws of comparative statics for generalized gross-substitute systems, and showed that essentially the same procedures as used in local alnalyses lead us to global results. The first and second Hicksian laws in the large in fact have a close relationship to the global 'weak' Le Chatelier-Samuelson principle (see Fujimoto (1980, Theorem 1)).

Although the problems in this paper refering to the global 'strong' principle, at first sight, seem to be different from the previous ones, and therefore require a distinct technique, we show that a similar argument using a homotopy is applicable. This suggests that the path-following approach may find applications to a wide variety of economic problems.

Appendix

In this appendix, we use the following notation.

1. a^i: the ith row vector of a matrix A.
2. $b(i)$: a vector obtained from a vector b by deleting the ith component.
3. $A\binom{i}{j}$: a matrix obtained from A by deleting the ith row and the jth column of A.
4. I: the set $\{1, \ldots, n\}$.
5. $\bar{J}$: the relative complement of J with respect to the set I.
6. $I(i)$: a subset of I by deleting the element i of I.
7. $J(i)$: a subset of I which does not contain i.
8. $\bar{J}(i)$: the relative complement of $J(i)$ with respect to the set $I(i)$.
9. $\bar{A}\binom{k}{h}$: a matrix obtained from $A\binom{k}{h}$ by bringing $a^h(h)$ in the place between $a^{k-1}(h)$ and $a^{k+1}(h)$ (see Uekawa (1971, p. 214)).
10. A_{kh}: the (k, h)th cofactor of a matrix A.

We first show the existence and uniqueness of a Cournot equilibrium.

Lemma A.1 *Let $h(x,t) \equiv (1-t)(x-\bar{x})+tf(x)$, where $\bar{x}$ denotes an interior point of Ω^N. Then, for any $x \in \Omega^N$ and any $t \in [0,1]$, D_xh is a P-matrix, a matrix having all principal minors positive. In particular, Df is everywhere a P-matrix in Ω^N.*

Proof. Define the (i,j)th element of D_xh by h_{ij}. Notice that under Assumptions 3.1.3 and 3.1.4, D_xh is a nonnegative square matrix, and that $h_{ii} > h_{ij} = h_{ik}, i \neq j \neq k$ for any $x \in \Omega^N$ and any $t \in [0,1]$.

Let J be a nonempty proper subset of I, and denote the numbers of the elements of J and $\bar{J}$ by $\sharp J$ and $\sharp \bar{J}$, respectively. Then, we have for any $x \in \Omega^N$ and any $t \in [0,1]$,

$$\frac{1}{\sharp J}\sum_{j \in J} h_{ij} > \frac{1}{\sharp \bar{J}}\sum_{j \in \bar{J}} h_{ij}, \qquad i \in J,$$

$$\frac{1}{\sharp J}\sum_{j \in J} h_{ij} < \frac{1}{\sharp \bar{J}}\sum_{j \in \bar{J}} h_{ij}, \qquad i \in \bar{J}.$$

It follows from Uekawa (1971, Theorem 1) that the transposed matrix of D_xh, and therefore D_xh itself becomes a P-matrix for any $x \in \Omega^N$ and any $t \in [0,1]$. □

Theorem A.1 *Suppose that Assumption 3.1 holds. Then, there exists uniquely a Cournot equilibrium in the interior of Ω^N.*

Proof. Using the homotopy defined in the above lemma, we can make sure that by Assumption 3.1.5, h is boudary-free at all $t \in [0,1]$.

According to Lemma A.1, D_xh is nonsingular for any $x \in \Omega^N$ and any $t \in [0,1]$. In addition, Df is everywhere a P-matirx in Ω^N. Therefore, we can construct a homotopy-path which starts from $(\bar{x},0)$ and terminates at $(x^*,1)$, where x^* is a unique solution to $f(x)=0$. □

The lemma below is used in the proof of Theorem 3.1.

Lemma A.2 *Let $A \equiv [a_{ij}]$ be a nonnegative (resp. positive) square matrix of order n, and J be any given nonempty proper subset of I. Suppose that for any* *given* k $(k \notin J)$, *there exist* $x_{Jj}^{(k)} > 0, j \in J$, *such that*

$$\sum_{j \in J} a_{ij} x_{Jj}^{(k)} > a_{ik}, \qquad i \in J, \tag{8}$$

$$\sum_{j \in J} a_{ij} x_{Jj}^{(k)} \leq a_{ik}, \qquad i \in \bar{J}, \tag{9}$$

with a strict inequality for $i = k$. *Then,* A *is a P-matrix. Moreover,* A^{-1} *has positive diagonal entries and nonpositive (resp. negative) off-diagonal ones.*

Proof. The proof is similar to that of sufficiency of Uekawa (1971, Theorem 5). We first note that, from the above inequalities, A has positive diagonal elements. Following Uekawa's procedure, we can then ascertain that A is a P-matrix (see Uekawa (1971, pp. 214–215)).

Furthermore, let $h \notin J$ and $h \neq k$. Put $x_{Jj} \equiv \sum_{k \in \bar{J}(h)} x_{Jj}^{(k)}$ if $j \in J(h)$, while if $j \in \bar{J}(h)$, put $x_{Jj} \equiv 1$. Then, summing (8) and (9) over $k \in \bar{J}(h)$, respectively, we arrive at the inequalities

$$\begin{aligned} \sum_{j \in J(h)} a_{ij} x_{Jj} &> \sum_{j \in \bar{J}(h)} a_{ij} x_{Jj}, \qquad i \in J(h), \\ \sum_{j \in J(h)} a_{ij} x_{Jj} &< \sum_{j \in \bar{J}(h)} a_{ij} x_{Jj}, \qquad i \in \bar{J}(h), \\ \sum_{j \in J(h)} a_{hj} x_{Jj} &\leq \sum_{j \in \bar{J}(h)} a_{hj} x_{Jj}. \end{aligned} \tag{10}$$

Therefore, det $\bar{A}\binom{k}{h} \geq 0$ using Uekawa (1971, lemma 5). In particular, if A is a positive matrix, let $x_{Jj} \equiv 1 + \epsilon$ for $j \in \bar{J}(h)$, where $\epsilon > 0$ is sufficiently small so that strict inequalities (10) remain valid. Then, applying Uekawa (1971, Theorem 1), we have det $\bar{A}\binom{k}{h} > 0$. Noting that $A_{kh} = -\det \bar{A}\binom{k}{h}$, we obtain the lemma since A is a P-matrix. □

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