Kansai University Review of Economics
No. 3 (March 2001), pp. 69 – 72

On the Existence of Equilibria for an *n*-Coalitions Game*

Hirokazu Sakane

A strategic behavior taken by many agents in a group is investigated. In our game model, *n*-coalitions are formed by many agents. The optimum coalition strategy is selected in accordance with a consensus of almost each agent which belongs to the coalition, given the other coalition's strategies. The state that almost each agent in every coalition no longer has any insentive to change its strategy is called equilibrium points. The purpose of this note is to present an existence theorem of equilibrium points for the game.

Keywords: Coalition Structures, Nash Equilibria

1. The Introduction

In this note, a strategic behavior taken by many agents in a group is investigated. It is assumed in our model that *n*-coalitions are formed and *many* agents which belong to each coalition behave in cooperation. Under these suppositions, the following three notions must be explained to define a game, that is, (i) *a set of coalitions*, (ii) *a strategy set for a coalition*, and (iii) *a payoff function for a coalition*. First, The set of coalitions has a meaning of a set of *players* in our model. It is also called a *coalition structure*, which is a partition of the set of agents. And it is fixed arbitrarily for the whole of the note. Second, A strategy set for a coalition must be defined besides a strategy set for an agent. It is supposed that the notion is defined as an aggregation of strategy sets for agents which belong to the coalition. An element of the strategy set is *a strategy for the coalition*. Finally, a payoff function for a coalition is a real valued function defined on the strategy set for the coalition, given the other coalition's strategies. A game is specified by these three notion. The game is called in this note *'n-coalitions game'*. In the game, it is supposed that the optimum coalition strategy which guarantees the maximum payoff for a coalition is selected in accordance with a consensus of almost each agent belonging to the coalition,

* This research was financially supported by the Kansai University Grant-in-Aid for the Faculty Joint Research Program, 2000.

given the other coalition's strategies. The state that almost each agent in every coalition no longer has any insentive to change its strategy is called equilibrium points in the n-coalitions game. The purpose of this paper is to present an existence theorem of equilibrium points for the game.

2. The Definitions

The space of agents is a measure space $(I, \mathscr{I}, \mu)$, where I is a set of agents, $\mathscr{I}$ is a σ-algebra of subsets of I and $\mu : \mathscr{I} \to R_+$ is a countably additive, nonatomic finite measure. It is supposed in our model that n-coalitions (which are subsets of I) are formed and each agent belongs to one of coalitions $\{C_\lambda\}_{\lambda=1}^{n}$ which satisfies the conditions $\cup_{\lambda=1}^{n} C_\lambda = I$ and $C_\lambda \cap C_\eta = \emptyset$ ($\lambda \neq \eta$). That is $\{C_\lambda\}_{\lambda=1}^{n}$ is a patition of I. It is also called a *coalition structure*. The coalition structure is fixed for the rest of the note. Since agents which belong to each coalition behave in cooperation, each coalition may be regarded as a unit of a decision maker. In a word, C_λ is considered to be a player for each $\lambda \in \{1, \cdots, n\}$.

(D.1) $\{C_\lambda\}_{\lambda=1}^{n}$ is the set of players, which satisfies the conditions $\cup_{\lambda=1}^{n} C_\lambda = I$ and $C_\lambda \cap C_\eta = \emptyset$ ($\lambda \neq \eta$); (The Coalition Structure).

The strategy set for agent $i \in I$ is denoted by $X(i)$ and it is assumed that $X(i) \subset R^\ell$. Since $X : I \to R^\ell$ is regarded as a set-valued map, the set of integrable selections of X may be defined by $\mathscr{L} := \{x \in L_1 | x(i) \in X(i), \mu\text{-a.e.}\}$. The strategy set for coalition $C_\lambda \in \mathscr{P}(I)$ is defined by $\int_{C_\lambda} X(i) d\mu := \{\int_{C_\lambda} x(i) d\mu \; | x \in \mathscr{L}\}$, and an element a_{C_λ} of the set is called a strategy for the coalition.

(D.2) $\int_{C_\lambda} X(i) d\mu := \{\int_{C_\lambda} x(i)\, d\mu | x \in \mathscr{L}\}$ ($\lambda \in \{1, \cdots, n\}$); (The Strategy Set for a Coalition).

It is supposed that each coalition C_λ selects an element of $\int_{C_\lambda} X(i) d\mu$ which guarantees the maximum payoff for C_λ, given the other coalition's strategies $(a_{C_1}, \cdots, a_{C_n})$. A payoff function for a coalition is a real-valued function defined as follows.

(D.3) $V_{C_\lambda}(\cdot, (a_{C_1}, \cdots, a_{C_n})) : \int_{C_\lambda} X(i) d\mu \to R_+$. ($\lambda \in \{1, \cdots, n\}$); (The Payoff Function for a Coalition).

The n-coalitions game is specified by the list of (D.1)~(D.3) as follows.

(D.4) $(\{C_\lambda\}_{\lambda=1}^n, \{\int_{C_\lambda} X(i)d\mu\}_{\lambda=1}^n, \{V_{C_\lambda}(\cdot, (a_{C_1}, \cdots, a_{C_n}))\}_{\lambda=1}^n)$; (The n-Coalitions Game).

(D.5) The equilibrium points for the n-coalitions game is the n-tuple $(a^*_{C_1}, \cdots, a^*_{C_n}) \in (\int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu)$ satisfing the condition $[\forall\lambda\in\{1, \cdots, n\}, V_{C_\lambda}(a^*_{C_\lambda}, (a^*_{C_1}, \cdots, a^*_{C_n})) = max_{a_{C_\lambda}\in\int_{C_\lambda} X(i)d\mu} V_{C_\lambda}(a_{C_\lambda}, (a^*_{C_1}, \cdots, a^*_{C_n}))]$; (The Equilibrium Points for the n-Coalitions Game).

3. The Existence of Equilibrium Points for The *n*-Coalitions Game

Now, we turn to prove a theorem on the existence of equilibrium points for the game $(\{C_\lambda\}_{\lambda=1}^n, \{\int_{C_\lambda} X(i)d\mu\}_{\lambda=1}^n, \{V_{C_\lambda}(\cdot, (a_{C_1}, \cdots, a_{C_n}))\}_{\lambda=1}^n)$

Theorem: Under the following conditions (i)~(iv), there are equilibrium points for $(\{C_\lambda\}_{\lambda=1}^n, \{\int_{C_\lambda} X(i)d\mu\}_{\lambda=1}^n, \{V_{C_\lambda}(\cdot, (a_{C_1}, \cdots, a_{C_n}))\}_{\lambda=1}^n)$,
(i) $X : I \to R^\ell$ satisfies the condition $[\forall x\in\mathscr{L}, \exists y\in L_1, \|x(i)\| \leq y(i)$, μ-a.e.],
(ii) $X(i)$ is closed in R^ℓ, μ-a.e.,
(iii) *graph* $(X) := \{(x(i), i)\in R^\ell\times I \mid x(i)\in X(i)\}$ is an element of $\mathscr{I}\times\mathscr{B}(R^\ell)$,
(iv) for any $\lambda\in\{1, \cdots, n\}$ and any $(a_{C_1}, \cdots, a_{C_n})\in(\int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu)$, the payoff function $V_{C_\lambda}(\cdot, (a_{C_1}, \cdots, a_{C_n})) : \int_{C_\lambda} X(i)d\mu \to R_+$ is quasi-concave.

Proof: Fix any $\lambda\in\{1, \cdots, n\}$. Let χ_{C_λ} be the characteristic function. Since the correspondence $\chi_{C_\lambda} X$ also satisfies the conditions (i), (ii) and (iii), it is shown that $\int \chi_{C_\lambda} X(i)d\mu = \int_{C_\lambda} X(i)d\mu$ is a non-empty compact set by Theorem 2 and 4 in Aumann (1965). Moreover, it is proved that $\int_{C_\lambda} X(i)d\mu$ is convex by Lemma A in Vind (1964). The best response correspondence is defined as follows,

$$(1) \quad \alpha_\lambda : \int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu \to \int_{C_\lambda} X(i)d\mu, (a_{C_1}, \cdots, a_{C_n}) \mapsto a^*_{C_\lambda};$$

$$V_{C_\lambda}(a^*_{C_\lambda}, (a_{C_1}, \cdots, a_{C_n})) = max_{a_{C_\lambda}\in\int_{C_\lambda} X(i)d\mu} V_{C_\lambda}(a_{C_\lambda}, (a_{C_1}, \cdots, a_{C_n})).$$

Fix any $(a_{C_1}, \cdots, a_{C_n})\in\int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu$. An upper semi-continuity of α_λ is shown by the standard method. A convexity of $\alpha_\lambda(a_{C_1}, \cdots, a_{C_n})$ is obvious under the condition (iv) and the convexity of $\int_{C_\lambda} X(i)d\mu$. And it is shown by Theorem 2 and 4 in Aumann (1965) that $\alpha_\lambda(a_{C_1}, \cdots, a_{C_n})$ is a non-empty compact set. These results are true for any $(a_{C_1}, \cdots, a_{C_n}) \in \int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu$ and any $\lambda \in \{1, \cdots, n\}$. Let α define as follows.

$$\text{(2)} \qquad \alpha: \int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu \to \int_{C_1} X(i)d\mu \times \cdots \times \int_{C_n} X(i)d\mu$$

$$(a_{C_1}, \cdots, a_{C_n}) \mapsto (\alpha_1(a_{C_1}, \cdots, a_{C_n}), \cdots, \alpha_n(a_{C_1}, \cdots, a_{C_n})).$$

The correspondense α has the same properties as α_λ. Therefore, by Kakutani's fixed point theorem, there exists an element satisfing the condition $(a^*_{C_1}, \cdots, a^*_{C_n}) \in \alpha(a^*_{C_1}, \cdots, a^*_{C_n})$. The element $(a^*_{C_1}, \cdots, a^*_{C_n})$ satisfies the condition in (D.5).

4. The Concluding Remark

To search a desirable element for a coalition is not always a rational behavior for almost each agent in the coalition. Suppose that a preference relation for each agent $i \in I$ is represented by a utility function $u(i, \cdot, a_{C_1}, \cdots, a_{C_n}) : X(i) \to R_+$. When each $i \in C_\lambda$ receives a transferable utility $v_\lambda(i, a_{C_1}, \cdots, a_{C_n})$ satisfing the condition $V_{C_\lambda}(a^*_{C_\lambda}, (a_{C_1}, \cdots, a_{C_n})) = \int_{C_\lambda} v_\lambda(i, a_{C_1}, \cdots, a_{C_n})d\mu$, i compares $v_\lambda(i, a_{C_1}, \cdots, a_{C_n})$ and $u(x^*(i), a_{C_1}, \cdots, a_{C_n})$, where x^* is the strategy satisfing the condition $\int_{C_\lambda} x^*(i, a_{C_1}, \cdots, a_{C_n})d\mu = a^*_{C_\lambda}$. If $v_\lambda(i, a_{C_1}, \cdots, a_{C_n}) \geqq u(x^*(i), a_{C_1}, \cdots, a_{C_n})$ is concluded, agent $i \in C_\lambda$ willing to stay the coalition C_λ. If it is not so, however, agent $i \in C_\lambda$ will wish to deviate from C_λ. Such the behavioral pattern may not be analyzed by the game model. This problem remains as a matter to be discussed further.

References

Aumann, R. (1965): "Integrals of Set-valued Functions," *Journal of Mathematical Analysis and Applications 12*, 1-12.

Vind, K. (1964): "Edgeworth-Allocations in an Exchange Economy with Many Traders," *International Economic Review 5*, 165-177.