REFINED SOLUTIONS OF OPTIMAL STOPPING GAMES FOR SYMMETRIC MARKOV PROCESSES

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Abstract

We find refined solutions without exceptional starting points of the three problems of the optimal stopping, the zero-sum stopping game (Dynkin's game) and the non-zero sum stopping game for a general symmetric Markov process under the absolute continuity condition on the transition function.

1. Introduction

For a symmetric Markov process M on a general state space X, the solution of an optimal stopping problem was identified by Nagai[6)] with a quasi continuous version of the solution of a variational inequality formulated in terms of the Dirichlet form associated with M. This was then successfully extended to the Dynkin game (a zero sum stopping game) by Zabczyk[8)] and to a non-zero sum stopping game by Nagai[7)] (see section 2).

In each of the three types of optimal stopping problems of a symmetric Markov process M however, certain sets N of zero capacity are involved as exceptional starting points of M. The aim of this paper is to refine in section 3 those statements in the cited papers by showing that they hold without any exceptional starting point under the assumption that the transition function of M is absolutely continuous with respect to the underlying measure m. The key step in our proof is to refine the arguments of Nagai[6)] by using a positive continuous additive functional of finite potential formulated by the first author[3)]. The absolute continuity assumption is satisfied by many important symmetric Markov processes including the multidimensional Brownian motion and symmetric stable processes.

Zabczyk's work[8)] on the Dynkin game is of basic importance and of potential applicability. For instance, it has been applied to solving a one-dimensional singular control problem in Fukushima-Taksar[4)]. In identifying the saddle point of the Dynkin game however, Zabczyk[8)] employed a well-known penalty method together with the Dirichlet space theory. In the last section of the present paper, we will simplify this part of his proof by showing that the penalty method can be dispensed with.

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2. Summary of Three Types of Stopping Problem

Let X be a locally compact separable metric space, m be an everywhere dense positive Radon measure on X and $M = (X_t, P_x)$ be an m-symmetric Hunt process on X. We assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of M on $L^2(X; m)$ is regular in the sense that $\mathcal{F} \cap C_0(X)$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_0(X)$, where $C_0(X)$ denotes the space of continuous functions on X with compact support. There have been several works[6),8),7)] on optimal stopping problems for M formulated in relation to the Dirichlet form $(\mathcal{F}, \mathcal{E})$.

In Nagai[6)], It was showed that the value function of the optimal stopping problem

$$w(x) = \sup_{\sigma} E_x[e^{-\alpha\sigma} g(X_\sigma)], \quad x \in X \backslash N,$$

is quasi-continuous version of the solution of the following variational inequality

$$w \geq g \quad m\text{−}a.e., \quad w \in \mathcal{F}, \quad \mathcal{E}_\alpha(w, u - w) \geq 0 \quad {}^\forall u \in \mathcal{F},\ u \geq g \quad m - a.e., \tag{1}$$

where g is a quasi-continuous function in $\mathcal{F}$ and N is an appropriate properly exceptional set. Moreover, it holds that

$$w(x) = E_x[e^{-\alpha\hat{\sigma}} g(X_{\hat{\sigma}})], \quad {}^\forall x \in X \backslash N, \quad \text{where} \quad \hat{\sigma} = \inf\{t > 0;\ w(X_t) = g(X_t)\}.$$

Zabczyk[8)] then extended Nagai's result to the zero-sum stopping game (Dynkin game) as follows: for the pay-off function

$$J_x(\tau, \sigma) := E_x[e^{-\alpha(\tau\wedge\sigma)}(h(X_\tau)I_{\tau\leq\sigma} + g(X_\sigma)I_{\tau>\sigma})], \quad {}^\forall x \in X \backslash N, \tag{2}$$

the value function

$$v(x) = \sup_{\sigma} \inf_{\tau} J_x(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_x(\tau, \sigma), \quad {}^\forall x \in X \backslash N, \tag{3}$$

is quasi-continuous solution of the variational inequality

$$g \leq v \leq h \quad m\text{−}a.e., \quad v \in \mathcal{F}, \quad \mathcal{E}_\alpha(v, u - v) \geq 0 \quad {}^\forall u \in \mathcal{F},\ g \leq u \leq h \quad m\text{−}a.e. \tag{4}$$

and moreover, the pair $(\hat{\tau}, \hat{\sigma})$ of the hitting times defined by

$$\hat{\tau} = \inf\{t > 0;\ v(X_t) = h(X_t)\}, \quad \hat{\sigma} = \inf\{t > 0;\ v(X_t) = g(X_t)\}, \tag{5}$$

is a saddle point of the game in the sense that

$$J_x(\hat{\tau}, \sigma) \leq J_x(\hat{\tau}, \hat{\sigma}) \leq J_x(\tau, \hat{\sigma}), \quad v(x) = J_x(\hat{\tau}, \hat{\sigma}), \quad x \in X \backslash N, \tag{6}$$

for any stopping times τ, σ. Here g, $h \in \mathcal{F}$ are quasi-continuous functions satisfying $g \leq h$ m−$a.e.$, and N is an appropriate properly exceptional set.

Nagai[7)] considered the following non-zero sum stopping game which is not necessarily an extension of the zero-sum stopping game. Namely, for the pay-off functions defined by

$$\begin{aligned} J_x^1(\tau_1, \tau_2) &= E_x[e^{-\alpha(\tau_1\wedge\tau_2)}(g_1(X_{\tau_1})I_{\tau_1\leq\tau_2} + h_1(X_{\tau_2})I_{\tau_2>\tau_1})], \\ J_x^2(\tau_1, \tau_2) &= E_x[e^{-\alpha(\tau_1\wedge\tau_2)}(g_2(X_{\tau_2})I_{\tau_2\leq\tau_1} + h_2(X_{\tau_1})I_{\tau_1>\tau_2})], \end{aligned} \qquad x \in X \backslash N,$$

and for the quasi-continuous solutions $(\tilde{u}_1, \tilde{u}_2)$ of the quasi-variational inequality

$$u_1 \geq g_1 \vee U_\alpha(h_1)_{B(u_2,g_2)} \ m\text{-}a.e., \quad \mathcal{E}_\alpha(u_1, v - u_1) \geq 0 \ \ {}^\forall v \geq g_1 \vee U_\alpha(h_1)_{B(u_2,g_2)} \ m\text{-}a.e.,$$
$$u_2 \geq g_2 \vee U_\alpha(h_2)_{B(u_1,g_1)} \ m\text{-}a.e., \quad \mathcal{E}_\alpha(u_2, v - u_2) \geq 0 \ \ {}^\forall v \geq g_2 \vee U_\alpha(h_2)_{B(u_1,g_1)} \ m\text{-}a.e., \tag{7}$$

it was shown[7)] that the pair (τ_1^*, τ_2^*) of hitting times defined by

$$\tau_i^* = \inf\{t > 0 : \ \tilde{u}_i(X_t) = g_i(X_t)\}, \quad i = 1, 2,$$

is under some hypotheses (see subsection 3.3) a Nash equilibrium point of the non-zero sum stopping game with pay-off functions J_x^1, J_x^2 in the following sense:

$$\begin{aligned} \tilde{u}_i(x) &= J_x^i(\tau_1^*, \tau_2^*), && {}^\forall x \in X\backslash N, \ i = 1, 2, \\ J_x^1(\tau_1^*, \tau_2^*) &\geq J_x^1(\tau_1, \tau_2^*), && {}^\forall x \in X\backslash N, \ {}^\forall \tau_1 : \ \textit{stopping time}, \\ J_x^2(\tau_1^*, \tau_2^*) &\geq J_x^2(\tau_1^*, \tau_2), && {}^\forall x \in X\backslash N, \ {}^\forall \tau_2 : \ \textit{stopping time}. \end{aligned}$$

Here, g_i, $h_i \in \mathcal{F}$ are quasi-continuous functions satisfying that $g_i \leq h_i$ $m\text{-}a.e.$, and N is an appropriate properly exceptional set. $U_\alpha(h_1)$ is the least α-potential majorizing h_1 and $U_\alpha(h_1)_{B(u_2,g_2)}$ is the α-reduced function of $U_\alpha(h_1)$ on the set $B(u_2, g_2) = \{x \in X : \ \tilde{u}_2 = \tilde{g}_2\}$, $\tilde{u}_2$, $\tilde{g}_2$ denoting the quasi-continuous versions. $U_\alpha(h_2)_{B(u_1,g_1)}$ is similarly defined.

3. Refined Solutions of Stopping Problems

Let X, m, $M = (X_t, P_x)$ and $(\mathcal{E}, \mathcal{F})$ be as in section 2. Denote by X_Δ the one point compactification of X. We extend any numerical function u on X to X_Δ by setting $u(\Delta) = 0$. In this section we assume the absolute continuity condition for the transition function p_t of M:

$$p_t(x, \cdot) \ll m, \tag{8}$$

for all $t > 0$ and $x \in X$.

We will fix an $\alpha > 0$. A universally measurable function f on X taking value in $[0, \infty]$ is called α-excessive if $f(x) \geq 0$ and $e^{-\alpha t} p_t f(x) \uparrow f(x)$, $t \downarrow 0$, for each $x \in X$. A function $f \in \mathcal{F}$ is said to be an α-potential if $\mathcal{E}_\alpha(f, g) \geq 0$ for any non-negative $g \in \mathcal{F}$. For any α-potential $f \in \mathcal{F}$, the pointwise limit $\hat{f}(x) = \lim_{t\downarrow 0} p_t f(x) (\leq \infty)$, $x \in X$, exists and we see that $\hat{f} = f$ m-a.e. and that $\hat{f}$ is α-excessive. $\hat{f}$ is called the α-excessive regularization of f.

3.1 The optimal stopping problem We assume that g is a finely continuous function on X such that $g \in \mathcal{F}$ and

$$|g(x)| \leq \varphi(x), \quad x \in X, \tag{9}$$

for some finite α-excessive function φ on X.

It is known that the variational inequality (1) admits a unique solution w which is actually the least α-potential majorizing the function g m-a.e.

Theorem 1. *Let w be the solution of* (1) *and $\hat{w}$ be the α-excessive regularization of w. Then $\hat{w}$ is finite and*

$$\hat{w}(x) = \sup_{\sigma} E_x[e^{-\alpha\sigma} g(X_\sigma)], \quad {}^{\forall}x \in X. \tag{10}$$

Moreover, if we let $\hat{\sigma} = \inf\{t > 0;\ w(X_t) = g(X_t)\}$, *then*

$$\hat{w}(x) = E_x[e^{-\alpha\hat{\sigma}} g(X_{\hat{\sigma}})], \quad {}^{\forall}x \in X. \tag{11}$$

Proof First we show that $\hat{w}$ is finite. By virtue of Theorem 2.2.1 and Lemma 2.3.2 of the book[5)], we see that $\varphi \wedge \hat{w}$ is an α-potential in $\mathcal{F}$. Since $\hat{w}$ is the smallest α-potential majorizing g, we obtain $\hat{w} \leq \varphi \wedge \hat{w} \leq \varphi$, $m-a.e.$ But both $\hat{w}$ and φ are α-excessive and $\hat{w}(x) \leq \varphi(x) < \infty$, $x \in X$, yielding the finiteness of $\hat{w}$.

Next we prove the inequality

$$\hat{w}(x) \geq g(x), \quad x \in X. \tag{12}$$

Since $w \geq g$ m-a.e. we have $p_t w(x) \geq p_t g(x)$, ${}^{\forall}x \in X$, $t > 0$, and consequently it suffices to show that

$$\lim_{t \downarrow 0} p_t g(x) = g(x), \quad x \in X. \tag{13}$$

Fix $x \in X$. Since $\varphi \geq 0$ and $\lim_{t\downarrow 0} E_x[\varphi(X_t)] = \lim_{t\downarrow 0} p_t\varphi(x) = \varphi(x) = E_x[\varphi(X_0)]$, the family of random variables $\{\varphi(X_t),\ t \in (0,1)\}$ is uniformly integrable with respect to P_x, and so is the family of random variables $\{g(X_t),\ t \in (0,1)\}$ on account of the assumption (9). Since $g(X_t)$ converges to $g(X_0)$ as $t \downarrow 0$ P_x-a.s., the $L^1(P_x)$-convergence takes place, yielding (13)

We now turn to the proof of (10) and (11). Since w is an α-potential, there exists a positive Radon measure μ of finite energy integral such that,

$$\mathcal{E}_\alpha(\hat{w}, f) = \int_X f(x)\mu(dx), \quad f \in \mathcal{F} \cap C_0(X). \tag{14}$$

Therefore, we have $\hat{w}(x) = R_\alpha\mu(x)$, ${}^{\forall}x \in X$, where $R_\alpha\mu(x)$ is defined by the integral $\int_X r_\alpha(x,y)\mu(dy)$ in terms of a suitable resolvent density $\{r_\alpha(x,y)\}$ and $R_\alpha\mu$ is known to be α-excessive (see Problem 4.2.1 of the book[5)]. In particular, $R_\alpha\mu(x)$ is finite for any $x \in X$. Hence μ is in the class[3)]

$$S_{01} = \{\mu :\ \text{positive Radon}, \iint r_1(x,y)\mu(dx)\mu(dy) < \infty,\ R_1\mu(x) < \infty,\ {}^{\forall}x \in X\}.$$

Therefore[3)], there exists a positive continuous additive functional (PCAF) A_t in the strict sense such that $\hat{w}(x) = E_x[\int_0^\infty e^{-\alpha t} dA_t]$ for all $x \in X$. By the strong Markov property, we have for any stopping time σ

$$\hat{w}(x) = E_x[\int_0^\sigma e^{-\alpha t} dA_t] + E_x[e^{-\alpha\sigma}\hat{w}(X_\sigma)], \tag{15}$$

which combined with (12) implies $\hat{w}(x) \geq E_x[e^{-\alpha\sigma}\hat{w}(X_\sigma)] \geq E_x[e^{-\alpha\sigma} g(X_\sigma)]$. Therefore

$$\hat{w}(x) \geq \sup_{\sigma} E_x[e^{-\alpha\sigma} g(X_\sigma)], \quad x \in X. \tag{16}$$

Finally we set $B = \{x \in X;\ \hat{w}(x) = g(x)\}$. Since (14) holds for any finely continuous (and hence quasi-continuous) function $f \in \mathcal{F}$, we have $\int_{B^c} (\hat{w}(x) - g(x))\mu(dx)$ $= \int_X (\hat{w}(x) - g(x))\mu(dx) = \mathcal{E}_\alpha(w, w - g)$, which must vanish because w is an α-potential while (1) holds for $u = g$. Therefore, we get $\mu(B^c) = 0$ by (12). Hence we get

$$E_x[\int_0^\infty e^{-\alpha t} I_{B^c}(X_t) dA_t] = R_\alpha(I_{B^c}\mu)(x) = 0, \quad x \in X.$$

Thus, for any stopping time $\sigma \leq \hat{\sigma}$,

$$0 \leq E_x[\int_0^\sigma e^{-\alpha t} dA_t] \leq E_x[\int_0^\infty e^{-\alpha t} I_{B^c}(X_t) dA_t] = 0, \quad x \in X.$$

Consequently, we are led to (11) by putting $\sigma = \hat{\sigma}$ in (15). (11) and (16) implies (10). □

3.2 The zero-sum stopping game Let $g,\ h \in \mathcal{F}$ be finely continuous functions such that for all $x \in X$,

$$g(x) \leq h(x),\ |g(x)| \leq \varphi(x),\ |h(x)| \leq \psi(x), \tag{17}$$

where $\varphi,\ \psi$ are some finite α-excessive functions. For arbitrary pair of stopping times (τ, σ), let J_x be the payoff function defined by

$$J_x(\tau, \sigma) := E_x[e^{-\alpha(\tau \wedge \sigma)}(h(X_\tau)I_{\tau \leq \sigma} + g(X_\sigma)I_{\tau > \sigma})], \quad x \in X.$$

By the above assumption, $J_x(\tau, \sigma)$ is finite. We consider the following condition: there exist finite α-excessive functions $v_1,\ v_2 \in \mathcal{F}$ such that, for all $x \in X$,

$$g(x) \leq v_1(x) - v_2(x) \leq h(x). \tag{18}$$

It is known that the variational inequality (4) admits a unique solution v.

Theorem 2. *Assume condition* (18). *There exists a finite finely continuous function* $\hat{v}$, $x \in X$, *satisfying the variational inequality* (4) *and the identity*

$$\hat{v}(x) = \sup_\sigma \inf_\tau J_x(\tau, \sigma) = \inf_\tau \sup_\sigma J_x(\tau, \sigma), \quad x \in X, \tag{19}$$

where σ, τ *range over all stopping times. Moreover, the pair* $(\hat{\tau}, \hat{\sigma})$ *defined by*

$$\hat{\tau} = \inf\{t > 0;\ \hat{v}(X_t) = h(X_t)\}, \quad \hat{\sigma} = \inf\{t > 0;\ \hat{v}(X_t) = g(X_t)\},$$

is the saddle point of the game in the sense that

$$J_x(\hat{\tau}, \sigma) \leq J_x(\hat{\tau}, \hat{\sigma}) \leq J_x(\tau, \hat{\sigma}), \quad x \in X, \tag{20}$$

for all stopping times τ, σ.

Let $\hat{\overline{v}}$ and $\hat{\underline{v}}$ be α-excessive regularizations of $\overline{v}$ and $\underline{v}$ which solve the variational inequalities (27) and (28) in §4, respectively. Then, (29) implies

$$v_1(x) \geq \hat{\overline{v}}(x), \quad v_2(x) \geq \hat{\underline{v}}(x), \quad x \in X. \tag{21}$$

Since $\lim_{t\downarrow 0} p_t g(x) = g(x)$, $\lim_{t\downarrow 0} p_t h(x) = h(x)$, $x \in X$, as we saw in the preceding subsection, we further have $\hat{\overline{v}} \geq \hat{\underline{v}} + g$, $\hat{\underline{v}} \geq \hat{\overline{v}} - h$. In particular $\hat{\overline{v}}$ and $\hat{\underline{v}}$ are finite, and the difference $\hat{v} = \hat{\overline{v}} - \hat{\underline{v}}$ is a finite finely continuous function on X. In view of Corollary to Proposition 1 of Zabczyk[8], $\hat{v}$ is the unique solution of the variational inequality (4) and satisfies

$$g(x) \leq \hat{v}(x) \leq h(x), \quad x \in X. \tag{22}$$

Proof of Theorem 2 By (21), we have

$$|\hat{\underline{v}} + g| \leq \hat{\underline{v}} + |g| \leq v_2 + \varphi, \quad |\hat{\overline{v}} - h| \leq \hat{\overline{v}} + |h| \leq v_1 + \psi,$$

and we can apply Theorem 1 in obtaining, for any $x \in X$,

$$\begin{aligned}
\hat{\overline{v}}(x) &= \sup_{\sigma} E_x[e^{-\alpha\sigma}(\hat{\underline{v}} + g)(X_\sigma)] = E_x[e^{-\alpha\hat{\sigma}}(\hat{\underline{v}} + g)(X_{\hat{\sigma}})], \\
\hat{\sigma} &= \inf\{t > 0;\ \hat{\overline{v}}(X_t) = (\hat{\underline{v}} + g)(X_t)\} = \inf\{t > 0;\ \hat{v}(X_t) = g(X_t)\}, \\
\hat{\underline{v}}(x) &= \sup_{\tau} E_x[e^{-\alpha\tau}(\hat{\overline{v}} - h)(X_\tau)] = E_x[e^{-\alpha\hat{\tau}}(\hat{\overline{v}} - h)(X_{\hat{\tau}})], \\
\hat{\tau} &= \inf\{t > 0;\ \hat{\underline{v}}(X_t) = (\hat{\overline{v}} - h)(X_t)\} = \inf\{t > 0;\ \hat{v}(X_t) = h(X_t)\}.
\end{aligned}$$

On account of the proof of Theorem 1, we have, for any stopping times $\sigma \leq \hat{\sigma}$ and $\tau \leq \hat{\tau}$,

$$\hat{\overline{v}}(x) = E_x[e^{-\alpha\sigma}\hat{\overline{v}}(X_\sigma)], \ \hat{\underline{v}}(x) = E_x[e^{-\alpha\tau}\hat{\underline{v}}(X_\tau)].$$

Since $\{e^{-\alpha t}\overline{v}(X_t)\}$ and $\{e^{-\alpha t}\underline{v}(X_t)\}$ are non-negative P_x-supermartingales for each $x \in X$, we get for arbitrary stopping times τ and σ, and $x \in X$,

$$\hat{\overline{v}}(x) \geq E_x[e^{-\alpha\sigma}\hat{\overline{v}}(X_\sigma)], \quad \hat{\underline{v}}(x) \geq E_x[e^{-\alpha\tau}\hat{\underline{v}}(X_\tau)].$$

Therefore, we obtain for each $x \in X$

$$\begin{aligned}
\hat{v}(x) &= \hat{\overline{v}}(x) - \hat{\underline{v}}(x) \leq E_x[e^{-\alpha(\hat{\sigma}\wedge\tau)}\hat{\overline{v}}(X_{\hat{\sigma}\wedge\tau})] - E_x[e^{-\alpha(\hat{\sigma}\wedge\tau)}\hat{\underline{v}}(X_{\hat{\sigma}\wedge\tau})] \\
&= E_x[e^{-\alpha(\hat{\sigma}\wedge\tau)}\hat{v}(X_{\hat{\sigma}\wedge\tau})] \leq E_x[e^{-\alpha(\hat{\sigma}\wedge\tau)}(g(X_{\hat{\sigma}})I_{\hat{\sigma}<\tau} + h(X_\tau)I_{\tau\leq\hat{\sigma}})] = J_x(\tau, \hat{\sigma}),
\end{aligned}$$

where we have used (22). Similarly, we have $\hat{v}(x) \geq J_x(\hat{\tau}, \sigma)$, $x \in X$.
Thus $J_x(\hat{\tau}, \sigma) \leq \hat{v}(x) \leq J_x(\tau, \hat{\sigma})$, $x \in X$, which implies (19) and (20).

□

Remark 1. We are unable to prove Theorem 2 without separability condition (18) for obstacles g, h. As was shown by Zabczyk[8], the solution v of the variational inequality (4) can be approximated by solutions v_n corresponding to obstacles satisfying the separability condition. But the convergence takes place in the Dirichlet form, which does not imply the pointwise convergence without exceptional set in general.

3.3 Non zero-sum stopping game We shall also present a refined statement for the non-zero sum stopping game. We omit the proof because it can be readily carried out based on Theorem 1.

We assume $g_i,\ h_i \in \mathcal{F}$ are finely continuous functions satisfying that

$$g_i(x) \le h_i(x), \quad |g_i(x)| \le \varphi_i(x), \quad |h_i(x)| \le \psi_i(x), \quad x \in X,$$

where $\varphi_i,\ \psi_i$ are finite α-excessive functions, $i = 1, 2$. For any pair of stopping times (τ_1, τ_2), let $J_x^1,\ J_x^2$ be the pay-off functions defined by

$$J_x^1(\tau_1, \tau_2) = E_x[e^{-\alpha(\tau_1 \wedge \tau_2)}(g_1(X_{\tau_1}) I_{\tau_1 \le \tau_2} + h_1(X_{\tau_2}) I_{\tau_2 < \tau_1})],$$
$$J_x^2(\tau_1, \tau_2) = E_x[e^{-\alpha(\tau_1 \wedge \tau_2)}(g_2(X_{\tau_2}) I_{\tau_2 \le \tau_1} + h_2(X_{\tau_1}) I_{\tau_1 < \tau_2})],$$

for all $x \in X$ respectively. By the above assumption, $J_x^i(\tau_1, \tau_2)$, $i = 1, 2$, are finite. We assume the following:

$$\{x \in X :\ \widehat{U_\alpha(g_i)} = g_i\} \subset \{x \in X :\ \widehat{U_\alpha(h_j)} = h_j\}, \quad i, j = 1, 2, \quad i \ne j. \tag{23}$$

It is known that the solutions of (7) are α-potentials in $\mathcal{F}$.

Theorem 3. *Let (u_1, u_2) be solutions of quasi-variational inequality* (7)*, and we define*

$$\tau_i^* = \inf\{t > 0 :\ \hat{u}_i(X_t) = g_i(X_t)\}, \quad i = 1, 2.$$

Then, under the condition (23)*, (τ_1^*, τ_2^*) is the Nash equilibrium point of the non-zero sum stopping game with pay-off functions $J_x^1,\ J_x^2$ in the following sense: for any $x \in X$,*

$$\tilde{u}_i(x) = J_x^i(\tau_1^*, \tau_2^*), \quad i = 1, 2,$$
$$J_x^1(\tau_1^*, \tau_2^*) \ge J_x^1(\tau_1, \tau_2^*), \quad {}^\forall \tau_1 :\ \textit{stopping time},$$
$$J_x^2(\tau_1^*, \tau_2^*) \ge J_x^2(\tau_1^*, \tau_2), \quad {}^\forall \tau_2 :\ \textit{stopping time}.$$

4. Alternative Proof of (6)

In this section, under the setting for the Dynkin game in section 2, we present a simplification of the proof of (6) given in Zabczyk[8]. Let $g,\ h \in \mathcal{F}$ be quasi-continuous functions satisfying the inequality

$$g(x) \le h(x) \quad q.e. \tag{24}$$

We note that, as solutions of (1) for $|g|$ and $|h|$, there exist quasi-continuous α-potentials φ and $\psi \in \mathcal{F}$ such that

$$|g(x)| \le \varphi(x), \quad |h(x)| \le \psi(x) \quad q.e. \tag{25}$$

The obstacles g, h are said to satisfy the separability condition if there exist α-potentials $v_1,\ v_2 \in \mathcal{F}$ such that

$$g \le v_1 - v_2 \le h \quad m\text{–}a.e. \tag{26}$$

Zabczyk[8] has proceeded to the proof of (6) based on the next two assertions.

1. Under the condition (26), there exists a pair $(\overline{v},\underline{v})\in\mathcal{F}\times\mathcal{F}$ satisfying the quasi-variational problem

$$\overline{v}\geq\underline{v}+g \quad \mathcal{E}_\alpha(\overline{v},u-\overline{v})\geq 0 \quad {}^\forall u\geq\underline{v}+g,\ u\in\mathcal{F}, \tag{27}$$

$$\underline{v}\geq\overline{v}-h \quad \mathcal{E}_\alpha(\underline{v},u-\underline{v})\geq 0 \quad {}^\forall u\geq\overline{v}-h,\ u\in\mathcal{F}. \tag{28}$$

They satisfy

$$v_1\geq\overline{v},\quad v_2\geq\underline{v}\quad m\text{–}a.e. \tag{29}$$

and the difference $v=\overline{v}-\underline{v}$ is the unique solution of the problem (4).

2. there exist the sequences $\{g_n\}_n$, $\{h_n\}_n$ of quasi-continuous functions satisfying following conditions: g_n increases to g q.e. and in $(\mathcal{E}_\alpha,\mathcal{F})$, h_n increases to h q.e. and in $(\mathcal{E}_\alpha,\mathcal{F})$, $g_n\leq h_n$ m-a.e. and each pair (g_n,h_n) satisfies the separability condition (26). Denote by v (resp. v_n) the solution of the variational problem (4) for the obstacles g, h (resp. g_n, h_n). Then $v_n\to v$ *in* $(\mathcal{E}_\alpha,\mathcal{F})$.

We are now ready to give an alternative proof of (6). We assume that v_n and v are quasi-continuous already. By taking a subsequence if necessary, we may assume that the convergence takes place quasi-uniformly, namely, there exists increasing sequence of closed sets $\{X_k\}$ with $\lim_{k\to\infty}\mathrm{Cap}(X-X_k)=0$ such that the following convergence is uniform on each X_k:

$$v_n(x)\to v(x),\quad g_n(x)\to g(x),\quad h_n(x)\to h(x). \tag{30}$$

We introduce the hitting times defined by $\tau_\gamma=\inf\{t>0:\ v(X_t)+\gamma\geq h(X_t)\}$,

$\sigma_\gamma=\inf\{t>0:\ v(X_t)-\gamma\leq g(X_t)\}$, $T_k=\inf\{t>0:\ X_t\in X\backslash X_k\}$, $\gamma>0$, $k=1,2,\cdots$.

We may assume that $P_x(T_k\uparrow\infty)=1$ *q.e.*.

It suffices to show that

$$v(x)\leq E_x[e^{-\alpha\tau}v(X_\tau)],\qquad v(x)\geq E_x[e^{-\alpha\sigma}v(X_\sigma)], \tag{31}$$

for arbitrary stopping times τ and σ such that $\sigma\leq\hat{\tau}$ and $\tau\leq\hat{\sigma}$ $P_x-a.e.$ and $x\in X\backslash N$, where $\hat{\sigma}$ and $\hat{\tau}$ are hitting times defined by (5) and N is a properly exceptional set. In fact, we get from (31),

$$v(x)=E_x[e^{-\alpha(\hat{\tau}\wedge\hat{\sigma})}v(X_{\hat{\tau}\wedge\hat{\sigma}})]=J_x(\hat{\tau},\hat{\sigma}).$$

By (31) again, we have for any stopping time σ

$$J_x(\hat{\tau},\sigma)=E_x[e^{-\alpha\hat{\tau}}h(X_{\hat{\tau}})I_{\hat{\tau}\leq\sigma}+e^{-\alpha\sigma}g(X_\sigma)I_{\sigma<\hat{\tau}}]\leq E_x[e^{-\alpha(\hat{\tau}\wedge\sigma)}v(X_{\hat{\tau}\wedge\sigma})]\leq v(x),$$

since $g\leq v$ *q.e.* In the same way, we have $J_x(\tau,\hat{\sigma})\geq v(x)$ for any stopping time τ.

By virtue of **1**, **2** and Nagai[6)], we have $v_n=\overline{v}_n-\underline{v}_n$ with

$$\overline{v}_n(x)=\sup_\sigma E_x[e^{-\alpha\sigma}(\underline{v}_n+g_n)(X_\sigma)]=E_x[e^{-\alpha\hat{\sigma}_n}(\underline{v}_n+g_n)(X_{\hat{\sigma}_n})],\quad q.e.\ x\in X,$$

$$\hat{\sigma}_n=\inf\{t>0;\ \overline{v}_n(X_t)=(\underline{v}_n+g_n)(X_t)\}=\inf\{t>0;\ v_n(X_t)=g_n(X_t)\},$$

$$\underline{v}_n(x) = \sup_{\tau} E_x[e^{-\alpha\tau}(\overline{v}_n - h_n)(X_\tau)] = E_x[e^{-\alpha\hat{\tau}_n}(\overline{v}_n - h_n)(X_{\hat{\tau}_n})], \quad q.e.\ x \in X,$$
$$\hat{\tau}_n = \inf\{t > 0;\ \underline{v}_n(X_t) = (\overline{v}_n - h_n)(X_t)\} = \inf\{t > 0;\ v_n(X_t) = h_n(X_t)\}.$$

Therefore, by a similar argument to that of Nagai[6)], we can find a properly exceptional set N such that, for all $\sigma \le \hat{\sigma}_n$ and all $\tau \le \hat{\tau}_n$, we have

$$\overline{v}_n(x) = E_x[e^{-\alpha\sigma}\overline{v}_n(X_\sigma)], \quad \underline{v}_n(x) = E_x[e^{-\alpha\tau}\underline{v}_n(X_\tau)],$$

for all $x \in X\backslash N$. Furthermore, for all initial states $x \in X\backslash N$, processes $\{e^{-\alpha t}\overline{v}_n(X_t)\}$ and $\{e^{-\alpha t}\underline{v}_n(X_t)\}$ are non-negative P_x-supermartingales. Thus, for arbitrary stopping times τ, σ, and $x \in X\backslash N$, we obtain

$$\overline{v}_n(x) \ge E_x[e^{-\alpha\sigma}\overline{v}_n(X_\sigma)], \qquad \underline{v}_n(x) \ge E_x[e^{-\alpha\tau}\underline{v}_n(X_\tau)].$$

Hence, for all $x \in X\backslash N$, $v_n(x) = \overline{v}_n(x) - \underline{v}_n(x)$ is dominated by

$$E_x[e^{-\alpha(\hat{\sigma}_n\wedge\tau)}\overline{v}_n(X_{\hat{\sigma}_n\wedge\tau})] - E_x[e^{-\alpha(\hat{\sigma}_n\wedge\tau)}\underline{v}_n(X_{\hat{\sigma}_n\wedge\tau})] = E_x[e^{-\alpha(\hat{\sigma}_n\wedge\tau)}v_n(X_{\hat{\sigma}_n\wedge\tau})].$$

In particular, we have for all $\tau \le \hat{\sigma}_n$

$$v_n(x) \le E_x[e^{-\alpha\tau}v_n(X_\tau)], \quad x \in X\backslash N.$$

Similarly, we have for all $\sigma \le \hat{\tau}_n$

$$v_n(x) \ge E_x[e^{-\alpha\sigma}v_n(X_\sigma)], \quad x \in X\backslash N. \tag{32}$$

Now let us prove (31). Take any $\gamma > 0$ and fix k. According to (30), there exists p such that for all $n \ge p$,

$$|v_n(x) - v(x)| < \frac{\gamma}{2}, \ |h_n(x) - h(x)| < \frac{\gamma}{2}, \quad x \in X_k.$$

If $t < \tau_\gamma$, then $v(X_t) + \gamma < h(X_t)$. So, for all $t < \tau_\gamma \wedge T_k \wedge \sigma$, we obtain

$$v_n(X_t) \le v(X_t) + \frac{\gamma}{2} < h(X_t) - \frac{\gamma}{2} < h_n(X_t).$$

Hence $\tau_\gamma \wedge T_k \wedge \sigma \le \hat{\tau}_n$ and we have by (32),

$$v_n(x) \ge E_x[e^{-\alpha(\tau_\gamma\wedge T_k\wedge\sigma)}v_n(X_{\tau_\gamma\wedge T_k\wedge\sigma})], \quad x \in X_k. \tag{33}$$

On the other hand, $|v| \le \varphi + \psi$ *q.e.* by (25), and we may assume that

$$|v_n(x)| \le \varphi(x) + \psi(x) + \frac{\gamma}{2}, \quad x \in X_k.$$

By Lebesgue convergence theorem, we can let $n \to \infty$ in (33) to obtain

$$v(x) \ge E_x[e^{-\alpha(\tau_\gamma\wedge T_k\wedge\sigma)}v(X_{\tau_\gamma\wedge T_k\wedge\sigma})], \quad x \in X_k. \tag{34}$$

Since for quasi-continuous function $u \in \mathcal{F}$, $\{e^{-\alpha T}u(X_T) : T$ is a stopping time$\}$ is P_x-uniformly integrable as was shown in Lemma 5 of Zabczyk[8)], we have by letting $k \to \infty$, $v(x) \ge E_x[e^{-\alpha(\tau_\gamma\wedge\sigma)}v(X_{\tau_\gamma\wedge\sigma})]$. Letting $\gamma \downarrow 0$, we obtain by the quasi-left continuity of X_t and the quasi-continuity of v, the inequality $v(x) \ge E_x[e^{-\alpha(\hat{\tau}\wedge\sigma)}v(X_{\hat{\tau}\wedge\sigma})]$, completing the proof of (31). □

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